QUANTUM SYSTEMS OBEYING A GENERALIZED EXCLUSION-INCLUSION PRINCIPLE

A.M. Scarfone[1]

Dipartimento di Fisica - Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino, Italy;
Istituto Nazionale di Fisica della Materia - Unitá del Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

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Abstract

The aim of this work is to describe, at the quantum level, a many body system obeying to a generalized Exclusion-Inclusion Principle (EIP) originated by collective effects, the dynamics, in mean field approximation, being ruled by a nonlinear Schrödinger equation. The EIP takes its origin from a nonlinear kinetic equation where the nonlinearities describe interactions of different physical nature.
The method starts from the study of the kinetic behavior of many particle system which results to be nonlinear because of the interaction among the particles and introduces an effective nonlinear potential $U_{\rm EIP}$ which permits us to simulate the true interactions governing the dynamics of the system. The power of the method is tested in the case of spatially homogeneous classical N-particles system. Its kinetic in the momenta space is described by a Markoffian process. A judicious generalization of the particle current, permits us to obtain at the equilibrium, a statistical distribution interpolating in a continuous way the well known quantum statistics (Fermi-Dirac or Bose-Einstein).
Systems with statistical behavior interpolating between the Bose-Einstein and the Fermi-Dirac were introduced fifty years ago. Up to now many have been the attempts to generalize quantum statistics. As we will discuss in the first chapter

[1]e-mail: scarfone@polito.it

many of these generalized statistics can be obtained by means of EIP by an appropriate generalization of the many particle current expression. The result of this approach is a nonlinear Schrödinger equation with a complex nonlinearity. Only the imaginary part of this nonlinearity is fixed by EIP, while the real one, is obtained by the requirement that the system is canonical. This permits us to include together within EIP other interactions of different physical origin.
Systems obeying to EIP can be used to describe a wide range of physical situations. In condensed matter we can find many topics where EIP concepts can be applied like, for instance, in superconductivity to describe the formation of vortex-like excitations or in superfluidity to describe the formation of Bose-Einstein condensates.

The plane of the work is the following. In the first chapter, after an introduction, we explain the origin of EIP. In the same chapter we summarize the fundamental mathematical tools used in the thesis: variational principle, Lagrangian and Hamiltonian formalism for systems with infinity degrees of freedom, symmetries and conservation quantities. In chapter II, EIP is obtained from an appropriate deformation of the quantum current. From all the possible deformations of the current we chose the simplest one to study the mean properties of systems obeying to EIP. By means of variational principle we obtain a canonical nonlinear Schrödinger equation. We study the Lagrangian and Hamiltonian formulation both in the standard ψ representation than in the hydrodynamic one. In chapter III we look at the symmetries of the nonlinear Schrödinger equation obeying to EIP. By means of Nöther theorem we obtain and discuss conserved quantities associated with the symmetries of EIP potential and show that the systems obeying to EIP are conservative. Only internal forces are introduced in the system by the presence of EIP potential. In the same chapter we consider an important class of nonlinear gauge transformations. These transformations are obtained in order to linearize the continuity equation. As a consequence, the transformed evolution equation will be again a nonlinear Schrödinger equation containing a purely real nonlinearity. In chapter IV EIP-Schrödinger equation is coupled in a minimal way to an abelian gauge field. The dynamics of the gauge fields is described by the Maxwell-Chern-Simons Lagrangian. In presence of Chern-Simons coupling the evolution of the system is developed in (2+1) space-time dimension so that the model can be used to describe planar phenomena. This situation, for instance, is realized in the layer between two semiconductors or between superconductor and normal conductor and in particular in High-T_c superconductors where Chern-Simons coupling is used to describe the anyonic behavior of the excitations. We show that in this case, the anyonic statistic behavior ascribed to the system by the presence of Chern-Simons Lagrangian is not destroyed by the presence of EIP potential. The following two chapters are devoted to the study of special solutions of evolution equations of the system. In chapter V we study

the solitary waves in neutral systems. Here we obtain the differential equation for the shape of the solitons, and study its mean properties. Using the nonlinear gauge transformations obtained in chapter III we deduce the effective potential responsible of the formation of the solitons. Applications on the Bose-Einstein condensation are given. In chapter VI we consider charge, static, self-dual vortex-like solutions where the dynamics of gauge fields is ruled by the Chern-Simons Lagrangian. The shapes of the vortices as well as its electric and magnetic fields are obtained by numerical integration of appropriate differential equations. The physical properties of this solution are matched to the corresponding solutions known in literature where EIP is absent. Conclusions are reported in the last chapter.

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Contents

Chapter I

Introduction to EIP

1.1 Considerations on quantum many body systems

Macroscopic systems are characterized by a high number of degrees of freedom that makes impossible to determine exactly their evolution. This is due to many reasons like, for instance, the large number of evolution equations that must be solved (one for each degree of freedom), the incomplete knowledge of the initial conditions, the approximate knowledge of the physical interaction and so on. It also true that for complex systems it is more important to have information about physical observable that represent mean quantities characterizing the system as a whole.

We deal with the evolution equations describing the dynamics of the mean values of these observable. Because of the high number of degrees of freedom, it is more convenient to study complex systems in the context of field theories. Fields carry infinite degrees of freedom and their dynamics is ruled out by partial differential equations. The self-interactions introduced by the collective effects of the many particle systems are in general described by nonlinear terms in the evolution equations.

An important question in the topics of many body system is the statistic behavior of its constituents. This question is even more important for a quantum system where concept of statistics plays an important role.

In this thesis we deal with a nonlinear theory describing in the mean field approximation an interacting many body system where the nonlinear interactions are introduced starting from the kinetics of the system. Therefore the topics of the nonlinear partial differential equations and the statistic behavior of a quantum many body system are important subjects of this thesis. Let us clarify these two arguments.

We begin with a review about quantum statistics. Since the early days of quantum mechanics, it was clear that, from the principle of undistinguishability, the statistical behavior of a collection of identical particles must be different from the classical one. The particle statistics determines the structure of the many body wave functions, that turn out to be completely symmetric under permutations of identical particles named bosons which obey to the Bose-Einstein statistics or completely antisymmetric under permutation of identical particles, now called fermions and obeying to the Fermi-Dirac statistics.
The statistics obeyed by fermions and bosons have many important implications in quantum mechanics. For instance, the Pauli exclusion principle, which gives null probability for two fermions to be in the same quantum state, takes its origin from the antisymmetric fermionic wave functions; it has consequences like the quality of emitted spectrum of the atoms or the stability of compact objects likes white dwarfs or neutron stars. On the contrary, the totally symmetric bosonic wave function has the effect of enhancing the probability for the bosons to occupy a quantum state if this one is already occupied. This effect is responsible of phenomena like superconductivity or superfluidity.
In the early fifty Green [1, 2] found that the principles of quantum mechanics allow two kinds of statistics called *para-Bose* and *para-Fermi* statistics [3]. The parastatistics of order p are defined as the identical particles statistics in three dimensional space under the restriction of a possible number of particles in the symmetric or antisymmetric state for the para-Fermi and para-Bose respectively. The case with order $p = 1$ corresponds to the ordinary Fermi and Bose statistics. The different cases can be described by trilinear commutation relations among the creation and annihilation operators [4]. The parastatistics was applied to subnuclear components like quarks [5] to solve, for example, the puzzle in the quantum number of the barionic resonance Δ^{++}.
The case of a particle described by a value of p not an integer has been studied in Ref. [6, 7], in order to take into account small violations of Pauli exclusion principle or Bose statistics. In Ref [8] this particle was called *paronic*. However, the corresponding quantum field theories for such a particle turns out to have negative norm states and, as a consequence, are not acceptable [9, 10]. This saga culminates with a recent study of infinite statistics without assumptions on the parameter p [11, 12], all representations of the symmetric group can occur. The particles obeying this type of statistics are called *quons* [11, 13, 14]. The quonic statistics are described by the q-deformed bilinear commutation relations:

$$a\,a^\dagger - q\,a^\dagger\,a = 1 \ , \tag{1.1.1}$$

where q is a C-number with $|q| \leq 1$ and according to the Fredenhagen theorem [10],

cannot be embedded in the local algebra of observable. Differently from the paronic statistics the quonic one has positive definite squared norms for state vectors but notwithstanding Greenberg succeeded only in the nonrelativistic quantum theory due to the locality problem for the infinity statistics.
Differently from the quonic statististics, the concept of q-deformed oscillators or *q-oscillators* [15, 16, 17] takes its origin from the concept of quantum groups [18]. q-oscillator, in spite of classic oscillators obeys to the q-deformed unitary algebra $SU_q(N)$:

$$a\,a^\dagger \mp q\,a^\dagger\,a = q^{\mp N} \ , \tag{1.1.2}$$

with $q \in C$ and N is the number operator functions of the annihilation and creation operator a and $a^\dagger$. In (1.1.2) the sign $-$ or $+$ is reported to the q-bosons or q-fermions, respectively. The q-oscillators can formally be defined in any dimensional space but then violate the fundamental axioms of quantum field theories in terms of the relation between spin and statistics.

The situation changes radically in many body systems, the dynamics being confined in two spatial dimensions. As it was discussed by Wilczek [19, 20], the statistic behavior of a system depends on the property of interchange of identical particles and is related to the topological property of the configuration space of a collection of identical particles. In more than two dimensions only two possibilities are present. Here the fundamental group is that of permutation which has two one-dimensional representations corresponding to completely symmetric (Bosons) or antisymmetric (Fermions) wave functions. Differently, in two spatial dimensions the permutation group is replaced by the braid group [21]: the spin is not quantized in integer or half integer value and particles obey any statistics interpolating between the Bose and Fermi ones. These particles are called *anyons*. Successively in Ref. [22] were achieved similar conclusions using a completely different method based on the study of the unitary representation of the current algebra and diffeomorfism group. Anyons can occur in many physical applications in those condensed matter systems that can be effectively regarded as two dimensional. For example, anyons can occur in fractional quantum Hall effect where collective excitations have been identified as localized quasi-particles of fractional charges, fractional spin and fractional statistics [23, 24, 25] and also they can occur in high temperature superconductors recently discovered [26].
Is not too difficult to show in a heuristic way how anyonic statistics born in two spatial dimensions. Let $\psi(1,\,2)$ be the function describing two identical particles and assume that when we move particle 2 around particle 1 by an angle $\Delta\,\varphi$ the wave function changes as:

$$\psi(1,\,2) \to \psi'(1,\,2) = e^{i\,\nu\,\Delta\varphi}\,\psi(1,\,2) \ , \tag{1.1.3}$$

acquiring a phase factor depending on a statistical parameter ν. Now we look at the exchange of the two particles. This can be realized in two ways (see figure 1.1): moves particle 2 around particle 1 by an angle $\Delta\varphi = \pi$ or at the opposite side, by an angle $\Delta\varphi = -\pi$. In the two cases the wave function acquires an extra phase $exp(\pm\pi\,\nu)$. It is easy to recognize that in more than two spatial dimensions the two paths are topologically equivalent, so identifying the two transformations we have the relation:

$$e^{i\,\pi\,\nu} = e^{-i\,\pi\,\nu} \; , \tag{1.1.4}$$

which is true only if $\nu = 0, 1$ modulo 2, corresponding, respectively, to the well known bosonic and fermionc statistics. In more than two dimensions there are no other possibilities. Differently, in two dimensions we can not deform with continuity the paths, one into the other. They are topologically and physically distinct operations. The Eq. (1.1.4) does not necessarily hold any more and the statistical parameter ν can be not shrunken to take the value 0 or 1. The anyonic statistics was obtained in a more rigorous way by Y. -S. Wu [27]. For a review on anyons see for example Ref. [28].
We have emphasized the anyonic statistics because in chapter IV and VI we discuss a particle system obeying an exclusion-inclusion principle, where the matter field is coupled to an abelian gauge field whose dynamics is described by means of Chern-Simons Lagrangian. As it was discussed in Ref. [19, 29, 30] the presence of the Chern-Simons gauge field confers the anyonic behavior to the system.

Another definition of generalized quantum statistics has been formulated by Haldane [31, 32], is based on the rate of the number of the available states in a system of fixed size decreasing as more particles are added to it. This statistics is called *exclusion statistics*. The statistics of Haldane is formulated without any reference to spatial dimensions of the system. In his formulation of exclusion statistics, Haldane defines a generalized Pauli exclusion principle introducing the dimension d_N of the Hilbert space for single particle states as a finite and extensive quantity that depends on the number N of particles contained in the system. The exclusion principle implies that the number of available single particle states decreases as the occupational number increases

$$\Delta\, d_N = -g\,\Delta\, N \; , \tag{1.1.5}$$

where g is the parameter that characterizes the complete or partial action of the exclusion principle and makes possible the interpolation between the Bose-Einstein ($g = 0$) and the Fermi-Dirac ($g = 1$) statistics. The relevance of the Haldane statistics is in its implication in the fractional quantum Hall effect and anyonic

physics, in the Calogero-Sutherland model [33, 34, 35] and in the Luttinger model [36, 37].

It is known [38] that the effects due to the statistics are imposed by the Pauli exclusion principle to a system of free fermions and can be simulated by a repulsive potential in the coordinate space. Analogously, free bosons can be submitted to an attractive potential. We refer to it as a *statistical potential*. The statistical potential will be a nonlinear function of the fields describing the system and its spatial derivatives.

Several nonlinear Schrödinger equations (NLSEs) have been studied in the past and recently, they are commonly used in many different fields of research in physics. The cubic equation [39], for instance, with the nonlinearity proportional to $\pm|\psi|^2$, has been used to study the dynamical evolution of a boson gas with δ-function pairwise repulsion or attraction, responsible of its anyonic-like behavior [40]. Recently, this equation has been used to describe the Bose-Einstein condensation [41, 42, 43, 44, 45] and the dynamics of two-dimensional radiating vortices [46]. The nonlinear term $|\psi|^2$ appears also in the Ginzburg-Landau model of the superconductivity [47], a phenomenon investigated also by means of the Eckhaus equation which is a NLSE with a nonlinearity of the type $|\psi|^2+\alpha\,|\psi|^4$ [48]. The same equation appears in superfluidity, where the properties of a gas of bosons interacting via a two-body attractive and three-body repulsive δ function inter-particle potential are investigated [49, 50]. The Eckhaus equation can describe nonlinear waves in optical fibers with a "normal" dependence of the refractive index on the light intensity [46]. Another important example where nonlinearities in the Schrödinger equation induce a statistical behavior is given by Schrödinger-Chern-Simons theory. In Ref. [51] it was shown that the gauge fields can be expressed as functions of the matter fields and therefore can be eliminated from the initial equation by transforming it into a highly nonlinear Schrödinger equation which describe the same anyonic system.
In literature we can find NLSEs with complex and derivative type nonlinearities involving the quantities $(\boldsymbol{\nabla}\rho)^2$, $\Delta\rho$, $\boldsymbol{j}\cdot\boldsymbol{\nabla}\rho$, $\boldsymbol{\nabla}\cdot\boldsymbol{j}$ [52, 53, 54] as, for instance, in the Doebner-Goldin equation associated with a certain unitary group representation and describing irreversible and dissipative quantum systems. NLSEs with nonlinearities involving the quantity $\mathbf{j}$ have been also introduced to study planar systems of particles with anyon statistics [55].
We will see in chapter II that the potential introduced by the EIP is complex and derivative in the field ψ and ψ^*.

Solution spectrum of nonlinear partial differential equations is much rich than the linear one. Generally, the solutions of a nonlinear PDE can be split in two distinct classes: in the first we have the solutions that can be obtained with the perturbative method. This is possible if the coupling constant of the nonlinearity is small. In the

other class we find the nonperturbative solutions, obtained integrating directly the nonlinear PDE. This solutions depend on the coupling constant of the nonlinearity and are divergent in the zero limit. So, it is not possible to go continuously from the solutions of perturbative class to the nonperturbative class. In this one we find the soliton solutions. They are wave packets that propagate freely asymptotically not changing their shape and velocity also after a collision. These were discovered in the last century by Russel, but their interest in physics was emphasized only thirty years ago. Solitons are solutions of nonlinear partial differential equation. For instance, if we take into account the dispersion relation of the linear Schrödinger equation we have:

$$\hbar\,\omega = \frac{\hbar^2\,\boldsymbol{k}^2}{2\,m}\ , \tag{1.1.6}$$

and looking for the group velocity $\boldsymbol{v}_g = d\boldsymbol{k}/d\omega$:

$$\boldsymbol{v}_g = \frac{\hbar\,\boldsymbol{k}}{m}\ , \tag{1.1.7}$$

which depends on the number wave $\boldsymbol{k}$. This means that each component in the wave packet propagates with a different velocity and therefore the modulation of the shape of the packet is not maintained. Responsible of this dispersion is the Laplacian operator appearing in the kinetic term. To avoid this effect a confining potential is required which makes the evolution equation nonlinear.

Soliton solutions were found in many fields, for example light impulse in the wave guide [56, 57, 58, 59], propagation in electric circuits and plasma waves [60]. Their particle behavior makes them interesting in the physics of elementary particles [61] like for instance the t'Hooft-Polyakov monopole [62, 63].

The question of the research of the solutions of nonlinear differential equations is one of the most important topics of mathematical physics of the last years. One of the most powerful is the inverse scattering method [64]. It is a canonical transformation in the action-angle fields in which the Hamiltonian of the system appears to be diagonalized. This transformation is a Fourier transformation plus a Laplace one. When an evolution equation is solved with this method, the Hamiltonian spectrum results decomposed in a continuous part given by the perturbative solutions plus a discrete contribution given by the soliton solutions. Equations solved with the inverse scattering method are named S-integrable because their solutions are expressed as function on a spectral parameter. The more easy S-integrable equation is of course the Schrödinger equation which is solved by means of Fourier transformation.

Finally a new class of integrable equations have made their appearance recently. These are called C-integrable because can be linearized by means of change of de-

pendent variables. In this category we can find for example the Burger equation [65] or the Ekhaus equation [66].

1.2 What is EIP

We present the generalized Exclusion-Inclusion Principle (EIP) in the configuration space. For a rigorous introduction of the EIP we remind to the reference [67, 68, 69, 70, 71, 72]. In the first part of this section the discussion is keep at a classic level.
The EIP takes its origin from a classical nonlinear kinetic that takes into account inhibition or enhancement of the particle transition probabilities in the phase space. We start by considering a Markoffian process in a D-dimensional phase space. If we identify the state of the system by a vector in the phase space and setting $\pi(t,\, \boldsymbol{u} \to \boldsymbol{v})$ the transition probability from the state $\boldsymbol{u}$ to the state $\boldsymbol{v}$, the evolution equation for the distribution function $n(t,\, \boldsymbol{v})$ can be written as

$$\frac{\partial\, n(t,\, \boldsymbol{v})}{\partial\, t} = \int \left[\pi(t,\, \boldsymbol{u} \to \boldsymbol{v}) - \pi(t,\, \boldsymbol{v} \to \boldsymbol{u})\right]\, d^D\boldsymbol{u} \ . \tag{1.2.1}$$

The exclusion-inclusion principle is introduced into the classical transition probabilities by means of an inhibition or an enhancement factor.
In Ref. [68] it was postulated the following expression of the transition probability:

$$\pi(t,\, \boldsymbol{v} \to \boldsymbol{u}) = r(t,\, \boldsymbol{v},\, \boldsymbol{v} - \boldsymbol{u})\, \phi[n(t,\, \boldsymbol{v})]\, \psi[n(t,\, \boldsymbol{u})] \ , \tag{1.2.2}$$

where $r(t,\, \boldsymbol{v},\, \boldsymbol{v} - \boldsymbol{u})$ is the transition rate, $\phi[n(t,\, \boldsymbol{v})]$ is a function depending on the occupational distribution at the initial state $\boldsymbol{v}$ and $\psi[n(t,\, \boldsymbol{u})]$ depends on the arrival state. The function $\phi(n)$ must obey the condition $\phi(0) = 0$ because the transition probability is equal to zero if the initial state is empty. Furthermore, the function $\psi(n)$ must obey the condition $\psi(0) = 1$ because, if the arrival state is empty, the transition probability is not modified. The classical linear case is obtained if we chose $\phi(n) = n$ and $\psi(n) = 1$.
In the following, for reason of simplicity, we consider an one-dimensional space, in the first neighbor interaction approximation.
If we consider an infinitesimal transition from the state v to the state $v + dv$, the transition rate can be defined as [68]

$$r(t,\, v,\, \pm dv)\, dv^2 = D(t,\, v) \pm \frac{1}{2} J(t,\, v)\, dv \ , \tag{1.2.3}$$

where $J(t,\, v)$ and $D(t,\, v)$ are the drift and diffusion coefficients, respectively. In the simple case in which in a time interval dt only one transition is allowed, Eq. (1.2.1)

becomes

$$\begin{aligned}\frac{\partial\, n(t,\, v)}{\partial\, t} &= \pi(t,\, v-dv\to v)+\pi(t,\, v+dv\to v)\\ &- \pi(t,\, v\to v-dv)-\pi(t,\, v\to v+dv)\ . \end{aligned} \tag{1.2.4}$$

The (1.2.4) is a balance equation between the particle coming in $v \pm dv \to v$ in the site v with respect to that coming out $v \to v \pm dv$. It describes in the continuum limit the discrete Markoffian process reported in figure 1.2.
Expanding the r.h.s. of Eq. (1.2.4) in powers of dv, up to second order, in the limit $dv \to 0$ and taking into account the transition rate defined in Eq. (1.2.3), we obtain the following generalized, nonlinear Fokker-Planck equation:

$$\begin{aligned}\frac{\partial\, n(t,\, v)}{\partial\, t} &= \frac{\partial}{\partial\, v}\left[\left(J(t,\, v)+\frac{\partial\, D(t,\, v)}{\partial\, v}\right)\phi(n)\,\psi(n)\right.\\ &+ \left. D(t,\, v)\left(\psi(n)\frac{\partial\,\phi(n)}{\partial\, v}-\phi(n)\frac{\partial\,\psi(n)}{\partial\, v}\right)\right]\ . \end{aligned} \tag{1.2.5}$$

This is a continuity equation for the distribution function $n = n(t,\, v)$

$$\frac{\partial\, n(t,\, v)}{\partial\, t}-\frac{\partial\, j(t,\, v,\, n)}{\partial\, v}=0\ , \tag{1.2.6}$$

where the particle current $j = j(t,\, v,\, n)$ is given by:

$$j=\left(J(t,\, v)+\frac{\partial\, D(t,\, v)}{\partial\, v}\right)\phi(n)\,\psi(n)+D(t,\, v)\left(\psi(n)\frac{\partial\,\phi(n)}{\partial\, v}-\phi(n)\frac{\partial\,\psi(n)}{\partial\, v}\right) \tag{1.2.7}$$

When the system reaches the equilibrium configuration, the current (1.2.7) must be equal to zero. In particular, if we consider Brownian particles, the drift and the diffusion coefficients are given by:

$$J=\gamma\, v\ , \qquad\qquad D=\frac{\gamma}{\beta\, m}\ , \tag{1.2.8}$$

with γ a dimensional constant and $\beta = 1/k_B\, T$ where k_B is the Boltzmann constant. In this situation, when $j = 0$ Eq. (1.2.7) can be viewed as a first order differential equation with solution:

$$\frac{\phi(n)}{\psi(n)}=e^{-\epsilon}\ , \tag{1.2.9}$$

where $\epsilon = \beta\,(E-\mu)$ and $E = m\,v^2/2$ is the kinetic energy. The integration constant μ is the chemical potential which can be evaluated by fixing the number of particles of the system.

We show now how we can select the functions $\phi(n)$ and $\psi(n)$ in order to obtain the equilibrium distribution of some of the statistical distributions described in the previous section.
If we make the choice:

$$\phi(n) = n \ , \qquad\qquad \psi(n) = 1 + \kappa\, n \ , \tag{1.2.10}$$

from (1.2.9) we obtain the following stationary distribution:

$$n = \frac{1}{e^{\epsilon} - \kappa} \ . \tag{1.2.11}$$

Here we recognize the well know Maxwell-Boltzmann (MB), Bose-Einstein (BE) and Fermi-Dirac (FD) distributions when we fix the value of the statistic parameter $\kappa = 0, \pm 1$ respectively. Moreover for $-1 < \kappa < 1$ we have fractional distributions interpolating between the BE and FD ones.
As a second example we pose:

$$\phi(n) = n\,(1-g\,n)^{\frac{1-g}{2}}\,[1+(1-g)\,n]^{\frac{g}{2}} \ , \tag{1.2.12}$$

$$\psi(n) = (1-g\,n)^{\frac{1+g}{2}}\,[1+(1-g)\,n]^{1-\frac{g}{2}} \ , \tag{1.2.13}$$

and obtain the distribution of Haldane's statistics:

$$n\,e^{\epsilon} = (1-g\,n)^{g}\,[1+(1-g)\,n]^{1-g} \ , \tag{1.2.14}$$

obtained in Ref. [32]. The statistical parameter g is defined in (1.1.5). Finally if we pose:

$$\phi(n) = [n]_q \ , \qquad\qquad \psi(n) = [1+\sigma\,n]_q \ , \tag{1.2.15}$$

where $[x]_q$ is defined as:

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \ , \tag{1.2.16}$$

we obtain the q-oscillator distribution:

$$e^{\epsilon} = \frac{\sinh[\eta\,(1+\sigma\,n)]}{\sinh(\eta\,n)} \ , \tag{1.2.17}$$

with $\eta = \log q$. The parameter $\sigma = \pm 1$ permits us to describe bosonic or fermionic q-oscillator respectively.

We can give other examples but it is now clear how to obtain different statistical distributions starting from a nonlinear kinetic.
In this thesis *we shall study, in mean field approximation, a canonical quantum system obeying to an exclusion-inclusion principle obtained by choosing $\phi(n) = n$ and $\psi(n) = 1 + \kappa\, n$.*
This choice is made because it is the most easy to treat and, as we have shown before, permits us to simulate a system with an equilibrium distribution interpolating between the BE and FD distribution.

Now we introduce EIP in a quantum system.
Let us start by looking at an exclusion-inclusion principle in the configuration space. We consider the classical stochastic Marcoffian process in a 3-dimensional space described by the following forward nonlinear Fokker-Planck equation (FPE):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}^{(+)} = 0 \ , \tag{1.2.18}$$

with

$$\boldsymbol{j}^{(+)} = \boldsymbol{u}^{(+)}\, \rho\, (1 + \kappa\, \rho) - D\, \boldsymbol{\nabla}\, \rho \ , \tag{1.2.19}$$

where $\rho = \rho(t,\, \boldsymbol{x})$ is the occupational number or particle distribution in the configuration space, $\boldsymbol{u}^{(+)}$ is the forward velocity and $\kappa \in I\!\!R$.
Beside the forward FPE (1.2.19) we must also consider the backward one with current:

$$\boldsymbol{j}^{(-)} = \boldsymbol{u}^{(-)}\, \rho\, (1 + \kappa\, \rho) + D\, \boldsymbol{\nabla}\, \rho \ . \tag{1.2.20}$$

The semisum of the forward and backward FPEs will give us

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \left[\boldsymbol{v}\, \rho\, (1 + \kappa\, \rho) \right] = 0 \ , \tag{1.2.21}$$

where $\boldsymbol{v}$ is the current velocity given by:

$$\boldsymbol{v} = \frac{1}{2} \left[\boldsymbol{u}^{(+)} + \boldsymbol{u}^{(-)} \right] \ . \tag{1.2.22}$$

We stress one more on the meaning of the current. We define the transition probability from the site $\boldsymbol{x}$ to $\boldsymbol{x}'$ as $\pi(t,\, \boldsymbol{x} \to \boldsymbol{x}') = r(t,\, \boldsymbol{x},\, \boldsymbol{x}')\, \rho(t,\, \boldsymbol{x})\, [1 + \kappa\, \rho(t,\, \boldsymbol{x}')]$ with $r(t,\, \boldsymbol{x},\, \boldsymbol{x}')$ the transition rate and the choice $\phi(\rho) = \rho$, $\psi = 1 + \kappa\, \rho$. The transition

probability depends on the particle population $\rho(t,\, \boldsymbol{x})$ of the starting point $\boldsymbol{x}$ and also on the population $\rho(t,\, \boldsymbol{x}')$ of the arrival point $\boldsymbol{x}'$. If $\kappa > 0$ the $\pi(t,\, \boldsymbol{x} \to \boldsymbol{x}')$ introduces an inclusion principle. In fact the population at the arrival point $\boldsymbol{x}'$ stimulates the transition and the transition probability increases linearly with $\rho(t,\, \boldsymbol{x}')$. In the case $\kappa < 0$ the $\pi(t,\, \boldsymbol{x} \to \boldsymbol{x}')$ takes into account the Pauli exclusion principle. If the arrival point $\boldsymbol{x}'$ is empty $\rho(t,\, \boldsymbol{x}') = 0$, the $\pi(t,\, \boldsymbol{x} \to \boldsymbol{x}')$ depends only on the population of the starting point. If the arrival site is populated $0 < \rho(t,\, \boldsymbol{x}') \leq \rho_{\rm max}$ the transition is inhibited. The range of values the parameter κ can assume is bounded by the condition that $\pi(t,\, \boldsymbol{x} \to \boldsymbol{x}')$ be real and positive as the $r(t,\, \boldsymbol{x},\, \boldsymbol{x}')$. Then we conclude that κ is limited from below by the condition $\kappa \geq -1/\rho_{\rm max}$.

In the Nelson picture [73, 74, 75], a quantum system can be viewed as a stochastic process where the particles are subjected to a Brownian diffusion with a coefficient $D = \hbar/2\, m$. The quantum system is described by the complex wave function $\psi = \psi(t,\, \boldsymbol{x})$ and the quantity $\rho = |\psi|^2$ is interpreted as the particle probability density. We make the ansatz:

$$\psi = \rho^{1/2}\, \exp\left(\frac{i}{\hbar} S\right) \;, \tag{1.2.23}$$

where the phase S is related to the velocity $\boldsymbol{v}$ as [75, 76]:

$$\boldsymbol{v} = \frac{\boldsymbol{\nabla}\, S}{m} \;. \tag{1.2.24}$$

Using Eqs. (1.2.23) and (1.2.24) it is immediate to obtain from Eq. (1.2.21) the expression for the quantum current:

$$\boldsymbol{j} = -\frac{i\,\hbar}{2\, m}\, (1 + \kappa\, \rho)\, (\psi^*\, \boldsymbol{\nabla}\psi - \psi\, \boldsymbol{\nabla}\psi^*) \;, \tag{1.2.25}$$

which we can rewrite also as:

$$\boldsymbol{j} = \frac{\boldsymbol{\nabla} S}{m}\, \rho\, (1 + \kappa\, \rho) \;, \tag{1.2.26}$$

which is our starting assumption.
In Ref. [77], by using Nelson stochastic quantization method, it was obtained the following NLSE:

$$i\,\hbar\, \frac{\partial\, \psi}{\partial\, t} = -\frac{\hbar^2}{2\, m}\, \Delta\, \psi + \Lambda(\rho,\, \boldsymbol{j})\, \psi + V\, \psi \;, \tag{1.2.27}$$

where:

$$\Lambda(\rho,\, \boldsymbol{j}) = W(\rho) + i\, \mathcal{W}(\rho,\, \boldsymbol{j}) \;, \tag{1.2.28}$$

is the complex nonlinearity introduced by the EIP, and the expression of the imaginary part is:

$$\mathcal{W}(\rho,\, \boldsymbol{j}) = -\kappa \, \frac{\hbar}{2\,\rho} \, \boldsymbol{\nabla} \left(\frac{\boldsymbol{j}\,\rho}{1+\kappa\,\rho} \right) \; . \tag{1.2.29}$$

The real part $W(\rho)$ depends drastically on the quantization method. In the picture of stochastic quantization it was obtained the form:

$$W(\rho) = \kappa \, \frac{\hbar^2}{4\,m} \left[\frac{\Delta\,\rho}{1+\kappa\,\rho} + \frac{2-\kappa\,\rho}{2\,\rho\,(1+\kappa\,\rho)^2} (\boldsymbol{\nabla}\,\rho)^2 \right] \; . \tag{1.2.30}$$

It can be shown that the system described by Eqs. (1.2.27), (1.2.29) and (1.2.30) admits a continuity equation (1.2.18) with the current given by Eq. (1.2.25).
We define in the following the *quantum system obeying to EIP* as a system whose dynamics is described by a NLSE admitting a continuity equation with the quantum current gave by Eq. (1.2.25).

We conclude this section by considering some examples where EIP can be usefully applied. In nuclear physics the correlation effects between pairs of nucleons, viewed as fermions, are quite relevant in the interpretation of experimental results. Similarly, the interactions among bosons are relevant in various nuclear modes (superfluid model, interacting boson model, mean field boson approximation) and allow the explanation of many collective nuclear properties. The interaction among the fermionic valence nucleons outside the core produces pairs of correlated nucleons that can be approximated as particles with a behavior intermediate between fermionic and bosonic ones. This nuclear state (quasideuteron state) can be viewed as a particles system that obeys to EIP. Recently, it was studied a semiclassical model of photofission in the quasideuteron energy region [78]. We described the quasideuteron state as a mixture of fermion and boson states, with a good agreement of our calculated photofission cross sections of several heavy nuclei and experimental results. Another example is the Bose-Einstein condensation. The condensation originates from an attraction of statistical nature (Bose-Einstein statistics) among the particles. In several papers the Bose-Einstein condensation is studied by means of a cubic NLSE [41, 42] which describes in mean field approximation an attractive interaction between two bodies. In place of the cubic and simplest interaction, other interactions can be considered as, for instance, the one introduced by EIP to simulate an attraction among the particles.
In condensed matter we can consider the problem of the hopping transport on a lattice of ionic conductors: two ions having the same charge cannot occupy the same site due to their natural electrostatic repulsion; also the motion of the couple

electron-hole in a semiconductor can be described by means of EIP. In fact, while electrons and hole are fermions, together can be considered excited states behaving differently from that of a fermion or a boson.

1.3 Mathematical background

In this section we summarize the canonical method used in the study of systems with infinite degrees of freedom whose dynamics is described by a partial differential equations.

Let us start with some definitions and notations. Let M to be a complex smooth manifold with dimension D mapped by the coordinates x_i with $i = 1, \cdots, D$. Let $\mathcal{F}$ be the algebra of the functionals on $M \to I\!\!R$ of the type: $\mathcal{F} \ni P = \int \mathcal{P}\, d^D x$. We call the quantity $\mathcal{P}$ functional density. Let $\psi(t, \boldsymbol{x})$ and $\psi^*(t, \boldsymbol{x})$ be two fields on M^2, we set $\Psi \equiv (\psi, \psi^*)$ the complex 2-vector field on $M \times M$. We assume that the 2-vector field Ψ vanishes quickly on the boundary of M.

We consider now a non relativistic canonical quantum system described by the Lagrangian density $\mathcal{L}[\psi] \equiv \mathcal{L}(\partial_t\psi^*, \partial_t\psi, [\psi^*], [\psi]) \in \mathcal{F}$ depending on the scalar field $\psi \in M$ and its derivatives with $\partial_t = \partial/\partial t$. Here and in the following we adopt the notation in square bracket to indicate the dependence from the spatial derivative of any order. Moreover, for our purpose, we deal with the case in which the spatial derivative in the Lagrangian density are introduced by $\boldsymbol{\nabla}$ operator. Thus, the following notations means:

$$\mathcal{L}([a]) \equiv \mathcal{L}(a, \boldsymbol{\nabla} a, \boldsymbol{\nabla}^2 a, \boldsymbol{\nabla}^3 a, \cdots) \ . \tag{1.3.1}$$

Posing the action functional:

$$\mathcal{A} = \int \mathcal{L}\, d^D x\, dt \ , \tag{1.3.2}$$

the evolution equations for the fields ψ and ψ^* can be obtained by a variational principle [79]:

$$\delta\, \mathcal{A} = 0 \ , \tag{1.3.3}$$

where the variation of a functional $F \in \mathcal{F}$ is a 2-vector defined as:

$$\delta\, F = \left(\frac{\delta\, F}{\delta\, \psi}, \frac{\delta\, F}{\delta\, \psi^*} \right) \ . \tag{1.3.4}$$

[2]Because the theory developed in this thesis is nonrelativistic, here and after t play the role of a parameter

The functional derivative can be defined by means of Euler operator:

$$\frac{\delta F}{\delta \psi} \equiv \mathrm{E}_{\psi}(\mathcal{A}) \ , \qquad\qquad \frac{\delta F}{\delta \psi^*} \equiv \mathrm{E}^*_{\psi^*}(\mathcal{A}) \ , \tag{1.3.5}$$

which it is given by:

$$\mathrm{E}_{\psi} = \sum_{n=0}^{\infty} \sum_{J} (-D)^J \frac{\partial}{\partial_J \psi} \ , \tag{1.3.6}$$

where the second sum is extended over multi-indices $J = (j_t, j_1, \cdots, j_D)$ with $0 \le j_i \le n$, $i = t, 1, \cdots, D$ and $j_t + \sum j_i = n$. Eq. (1.3.3) are the Euler-Lagrange equations for the field ψ^* and ψ respectively. From Eq. (1.3.3) we obtain:

$$\sum_{n=0}^{\infty} (-1)^n \, \boldsymbol{\nabla}^n \frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla}^n \psi)} - \frac{d}{d\,t} \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} = 0 \ , \tag{1.3.7}$$

and its conjugate equation.
It is possible to show that the Euler operator satisfies the following property:

$$\mathrm{E}(\frac{\partial B}{\partial t} + \boldsymbol{\nabla}\, C) = 0 \ , \tag{1.3.8}$$

with $B,\, C \in \mathcal{F}$. Therefore, Lagrangian density which are total derivatives does not gives contribute to the evolution equations. It are named *null Lagrangian*.
In this thesis, we will consider canonical systems described by the following class of Lagrangian density in (3+1) dimensions:

$$\mathcal{L} = i\, \frac{\hbar}{2} \left(\psi^* \frac{\partial \psi}{\partial\, t} - \psi \, \frac{\partial \psi^*}{\partial\, t} \right) - \frac{\hbar^2}{2\,m} |\boldsymbol{\nabla}\psi|^2 - U([\psi^*],\, [\psi]) - V\, \psi^*\, \psi \ , \tag{1.3.9}$$

where V describes an external potential and $U([\psi^*],\, [\psi])$ is a real nonlinear potential. Using the Lagrangian (1.3.9) we obtain the following NLSE for the field ψ:

$$i\,\hbar\, \frac{\partial\, \psi}{\partial\, t} = -\frac{\hbar^2}{2\,m}\, \Delta\, \psi + \sum_{n=0}^{\infty} (-1)^n \, \boldsymbol{\nabla}^n \frac{\partial\, U}{\partial(\boldsymbol{\nabla}^n \psi)} + V\, \psi \ , \tag{1.3.10}$$

which is a derivative NLSE.

In a canonical theory, we can express the motion equations also in the Hamilton formalism. This requires that functional space M is a *Poisson manifold*, i.e. a D-dimensional smooth manifold equipped with the Poisson brackets. This one is defined as:

$$\{F,\, G\} = \int (\delta F\, \mathcal{D}\, \delta G)\, d^D x \ , \tag{1.3.11}$$

where $F, G \in \mathcal{F}$. $\mathcal{D}$ is the Hamiltonian linear operator acting on $\mathcal{F}$ which might depends by ψ and its derivative.
The Poisson brackets must satisfy the following properties:

- Linear
- Skew-symmetric
- Jacobi identity

This property is reflected on the definition of $\mathcal{D}$. The first property is evident from the linearity of $\mathcal{D}$ and the definition of Poisson brackets, the second require $\mathcal{D}$ to be skew-adjoint: $\mathcal{D}^\dagger = -\mathcal{D}$. Finally, the third condition is:

$$\{\{P, Q\}, R\} + \text{cyclic permutations} = 0 \; . \tag{1.3.12}$$

This last relation is true if $\mathcal{D}$ is a constant quantity not dependent on the fields ψ and ψ^*.
Eqs. (1.3.7) can be expressed in the Hamiltonian formalism if a functional $\mathcal{F} \ni H = \int \mathcal{H} \, d^D x$ called Hamiltonian function exists ($\mathcal{H}$ is the Hamiltonia density) so that:

$$\frac{\partial \Psi}{\partial t} = \mathcal{D} \, \delta \, H \; , \tag{1.3.13}$$

which is equivalent to the following expression in terms of Poisson brackets:

$$\frac{\partial \Psi}{\partial t} = \{\Psi, H\} \; . \tag{1.3.14}$$

In general, given a functional $F \in \mathcal{F}$ describing a physical observable of a system with Hamiltonian H, its time evolution can be written as:

$$\frac{d F}{d t} = \{F, H\} + \frac{\partial F}{\partial t} \; . \tag{1.3.15}$$

We remark *en passant* that for a system with finite dimension, the Poisson brackets must satisfy one more condition: it must be a derivative operator which means to satisfy the Leibnitz rule $\{A\,B, C\} = A\,\{B, C\} + \{A, C\}\,B$. For infinitely dimensional systems this condition is not imposed because the multiplication between elements of $\mathcal{F}$ is not well defined. In fact, it gives two functionals $A, B \in \mathcal{F}$, their product is not expressible as integral of a density functional i.e. the relation:

$$\int \mathcal{A} \, d^D x \int \mathcal{B} \, d^D x = \int \mathcal{C} \, d^D x \; , \qquad \text{for same density } \mathcal{C} \tag{1.3.16}$$

is not generally satisfied.
Let us now introduce the fields π_{ψ} and π_{ψ^*}, canonically conjugate momenta of the fields ψ and ψ^*, respectively and define:

$$\pi_{\psi} = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} , \qquad \qquad \pi_{\psi^*} = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi^*)} . \tag{1.3.17}$$

We can write the Hamiltonian density $\mathcal{H}([\psi], [\pi_{\psi}])$ related to the Lagrangian density by the Legendre transformation [80]):

$$\mathcal{H} = \pi_{\psi} \frac{\partial \psi}{\partial t} + \pi_{\psi^*} \frac{\partial \psi^*}{\partial t} - \mathcal{L} , \tag{1.3.18}$$

and the evolution of the system is described by the equations:

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \frac{\delta H}{\delta \pi_{\psi}} , \\ \\ \frac{\partial \pi_{\psi}}{\partial t} &= -\frac{\delta H}{\delta \psi} . \end{aligned} \tag{1.3.19}$$

It is easy to see that Eqs. (1.3.19) are equivalent to Eq. (1.3.13) if we make the choice for the Hamiltonian operator:

$$\mathcal{D} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{1.3.20}$$

which is skew-hermitian and satisfies the Jacobi identity because it has constant entry.
With this definition for the operator $\mathcal{D}$ the expression (1.3.15) becomes:

$$\begin{aligned} \frac{d F}{d t} &= \int \left[\frac{\delta F(\mathbf{x})}{\delta \psi(\mathbf{z})} \frac{\delta H(\mathbf{y})}{\delta \pi_{\psi}(\mathbf{z})} - \frac{\delta H(\mathbf{y})}{\delta \psi(\mathbf{z})} \frac{\delta F(\mathbf{x})}{\delta \pi_{\psi}(\mathbf{z})} \right] d^D z \\ &+ \int \left[\frac{\delta F(\mathbf{x})}{\delta \psi^*(\mathbf{z})} \frac{\delta H(\mathbf{y})}{\delta \pi_{\psi^*}(\mathbf{z})} - \frac{\delta H(\mathbf{y})}{\delta \psi^*(\mathbf{z})} \frac{\delta F(\mathbf{x})}{\delta \pi_{\psi^*}(\mathbf{z})} \right] d^D z + \frac{\partial F}{\partial t} , \end{aligned} \tag{1.3.21}$$

which can be simplified following the Dirac procedure [81] (see sections 2.1 and 4.2). An important fact, in canonical theories, follows from the Nöther theorem [82], which states the link between symmetries for the Lagrangian (or Hamiltonian) and the conserved physical quantities related to it. We resume briefly the general method.

The Nöther theorem says that for each one-parameter symmetry group of the system, there is a physical observable related to it:

$$Q = \int \mathcal{J}^0 \, d^D x \ , \tag{1.3.22}$$

which is conserved as a consequence of the continuity equation:

$$\frac{\partial \mathcal{J}^0}{\partial t} + \frac{\partial \mathcal{J}^i}{\partial x_i} = 0 \ , \tag{1.3.23}$$

where $i = 1, \cdots, D$ (sum on i is assumed) and $\mathcal{J}^i$ are the components of a D-vector describing the flux density associated to the $\mathcal{J}^0$.
In fact, let $\delta \psi$ be the variation produced by the action of a group transformation on ψ, it generates a symmetry for the system if the action (1.3.2) does not change. This implies for the Lagrangian:

$$\delta \mathcal{L} = \partial_\nu f^\nu \ , \tag{1.3.24}$$

which says that it might change for a total divergence (null Lagrangian). To give an example, we consider the easy case in which only the first derivative of the fields are present in the Lagrangian density. Computing the variation in $\mathcal{L}$ we obtain:

$$\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi)} \delta \partial_\nu \psi + c.c. = \partial_\nu f^\nu \ , \tag{1.3.25}$$

and after integration by part, taking into account the motion equations (1.2.21) we arrive at Eq. (1.3.23) where:

$$\mathcal{J}^\nu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \psi\right)} \delta \psi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \psi^*\right)} \delta \psi^* - f^\nu \ . \tag{1.3.26}$$

We remember that the $\mathcal{J}^\nu$ is not univocally defined. In fact, we can add to the expression (1.3.26) the gradient of a skew-symmetric 2-tensor:

$$\mathcal{J}^\nu \to \widetilde{\mathcal{J}}^\nu = \mathcal{J}^\nu + \partial_\mu T^{\mu\nu} \ , \tag{1.3.27}$$

with

$$T^{\mu\nu} = -T^{\nu\mu} \ . \tag{1.3.28}$$

As a consequence of the skew-symmetry the new vector $\widetilde{\mathcal{J}}^\nu$ satisfies the same continuity equation (1.3.23) and the conserved quantities are left unchanged by the substitution (1.3.27) if the field goes to zero on the boundary of M.

Chapter II

Canonical Systems Obeying to the Exclusion-Inclusion Principle

In this chapter we introduce a class of canonical nonlinear Schrödinger equations obeying to a generalized inclusion-exclusion principle (EIP). This is accomplished trough an opportune deformation of the expression of the quantum current of the matter field ψ. The Lagrangian and Hamiltonian structure are studied both in the ψ-representation and in the hydrodynamic one. The approach used to include the EIP in the evolution equation does not determine in a unique way the form of the nonlinear potential. In fact, we show that in the nonlinear potential $U_{\rm EIP}$, which is a complex functional, its real part is not determinable from kinetic considerations. We impose the canonicity of the system in order to determine the real part of the nonlinear potential. The final result is different from those obtained by performing other quantization method. Moreover, the canonicity required leave us the possibility to include an arbitrary real nonlinear potential $U[\rho]$. This arbitrarity can be used to introduce other interactions acting on the system simultaneously with the EIP. Finally we study the mean property of the system. In particular we analyze the Eherenfest relations for observables like energy, linear and angular momentum.

2.1 Canonical systems

We start from the general expression of a nonlinear Schrödinger Lagrangian in $(D+1)-$ dimension of the form:

$$\mathcal{L} = i\,\frac{\hbar}{2}\left(\psi^*\,\frac{\partial\,\psi}{\partial\,t} - \psi\,\frac{\partial\,\psi^*}{\partial\,t}\right) - \frac{\hbar^2}{2\,m}\,|\boldsymbol{\nabla}\psi|^2 - U([\psi^*],\,[\psi]) - V\,\psi^*\,\psi\ , \qquad (2.1.1)$$

where $\boldsymbol{\nabla} = (\partial/\partial\, x_1,\, \partial/\partial\, x_2,\, \ldots \partial/\partial\, x_D)$ is the gradient operator in D-dimension, $\hbar$ is a constant with the dimension of an action that we identify with the Planck constant and m is the mass parameter.
The first two terms in the Lagrangian density are the same encountered in the standard linear quantum description, the quantity V is a potential describing external interaction. $U([\psi^*],\, [\psi])$ is the nonlinear term which we assume to be an analytic smooth functional of the fields ψ, ψ^* and their spatial derivatives. We assume U real to make the system not dissipative.
Using the Lagrangian we introduce the action:

$$\mathcal{A} = \int \mathcal{L}\, d^D x\, dt \ . \tag{2.1.2}$$

Applying the Euler operator E_{ψ^*}, defined in Eq. (1.3.6), to Eq. (2.1.2), we obtain the following NLSE for the field ψ:

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t} = -\frac{\hbar^2}{2\,m}\Delta\,\psi + \frac{\delta}{\delta\,\psi^*}\,U([\psi^*],\,[\psi]) + V\,\psi \ , \tag{2.1.3}$$

where $(\Delta = \partial^2/\partial\,x_1^2 + \partial^2/\partial\,x_2^2 + \cdots + \partial^2/\partial\,x_D^2)$ is the D-dimensional Laplacian operator.
To obtain the expression of the nonlinear potential $U([\psi^*],\,[\psi])$ in order to include the EIP in the system it is convenient to consider the Bohm-Madelung representation for the wave function ψ:

$$\psi(t, 1\,\boldsymbol{x}) = \rho(t,\,\boldsymbol{x})^{1/2}\,\exp\left[\frac{i}{\hbar}S(t,\,\boldsymbol{x})\right] \ . \tag{2.1.4}$$

The hydrodynamic fields, density of particles ρ and phase S [83] are related with ψ by means of:

$$\rho = |\psi|^2 \ , \tag{2.1.5}$$

$$S = i\,\frac{\hbar}{2}\,\log\left(\frac{\psi^*}{\psi}\right) \ . \tag{2.1.6}$$

The nonlinear potential $U([\rho],\,[S])$ becomes now a real functional of the fields ρ, S and their spatial derivatives.
By taking into account the Leibnitz rule for the functional derivative:

$$\frac{\delta}{\delta\,\psi^*} = \frac{\delta\,\rho}{\delta\,\psi^*}\,\frac{\delta}{\delta\,\rho} + \frac{\delta\,S}{\delta\,\psi^*}\,\frac{\delta}{\delta\,S} = \psi\,\frac{\delta}{\delta\,\rho} + i\,\frac{\hbar}{2\,\rho}\,\psi\,\frac{\delta}{\delta\,S} \ , \tag{2.1.7}$$

Eq. (2.1.3) is rewritten in the form:

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t}=-\frac{\hbar^2}{2\,m}\Delta\,\psi+\left[\frac{\delta}{\delta\,\rho}\,U([\rho],\,[S])\right]\,\psi+i\,\frac{\hbar}{2\,\rho}\,\left[\frac{\delta}{\delta\,S}\,U([\rho],\,[S])\right]\,\psi+V\,\psi\ . \tag{2.1.8}$$

Introducing the nonlinear quantities:

$$W([\rho],\,[S])=\frac{\delta}{\delta\,\rho}\,U([\rho],\,[S])\ , \tag{2.1.9}$$

$$\mathcal{W}([\rho],\,[S])=\frac{\hbar}{2\,\rho}\,\frac{\delta}{\delta\,S}\,U([\rho],\,[S])\ , \tag{2.1.10}$$

the NLSE (2.1.8) becomes:

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t}=-\frac{\hbar^2}{2\,m}\Delta\,\psi+W([\rho],\,[S])\,\psi+i\,\mathcal{W}([\rho],\,[S])\,\psi+V\,\psi\ . \tag{2.1.11}$$

Using this expression it is easy to obtain the continuity equation of the system. By taking the product of (2.1.11) times ψ^* and subtract the complex conjugate form, we arrive to the expression:

$$\frac{\partial\,\rho}{\partial\,t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}_0=\frac{2}{\hbar}\,\rho\,\mathcal{W}([\rho],\,[S])\ , \tag{2.1.12}$$

where

$$\boldsymbol{j}_0=-\frac{i\,\hbar}{2\,m}\,(\psi^*\,\boldsymbol{\nabla}\psi^*-\psi\,\boldsymbol{\nabla}\psi^*)\ , \tag{2.1.13}$$

is the standard quantum current density.

Following the discussion made in section 1.2, we require that the system described by (2.1.11) admits the following expression for the quantum current [cfr. Eq. (1.2.25)]:

$$\boldsymbol{j}=(1+\kappa\,\rho)\,\boldsymbol{j}_0\ , \tag{2.1.14}$$

and that the continuity equation (2.1.12) becomes:

$$\frac{\partial\,\rho}{\partial\,t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0\ . \tag{2.1.15}$$

This can be accomplished if, taking into account expression (2.1.14) and (2.1.12), we select for the term $\mathcal{W}$ the form:

$$\mathcal{W}([\rho],\,[S])=-\kappa\,\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\cdot(\boldsymbol{j}_0\,\rho)=-\kappa\,\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{j}\,\rho}{1+\kappa\,\rho}\right)\ , \tag{2.1.16}$$

which has the same form of Eq. (1.2.29) obtained in Ref. [77] using the stochastic quantization method. Eq. (2.1.16) can be expressed with the hydrodynamic fields as:

$$\mathcal{W}([\rho],\,[S]) = -\kappa\,\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla} S}{m}\,\rho^2\right)\ , \tag{2.1.17}$$

and taking into account the dependence of $\mathcal{W}$ from the nonlinear potential $U([\rho],\,[S])$ given by Eq. (2.1.10) we obtain:

$$U([\rho],\,[S]) = \kappa\,\frac{(\boldsymbol{\nabla} S)^2}{2\,m}\,\rho^2 + \widetilde{U}([\rho])\ , \tag{2.1.18}$$

which is defined modulo a real functional of the field ρ and its spatial derivative. We note that starting from the continuity equation only the dependence of $U([\rho],\,[S])$ from the phase can be determinate whilst the dependence from the field ρ is not fixed which means that an arbitrary real quantity $\widetilde{U}([\rho],\,[S])$ functional of ρ, can be added compatibly with the EIP, without removing the canonicity of the system. Hereinafter we call "*EIP potential*" the quantity $U_{\rm EIP}(\rho,\,S)$ given by:

$$U_{\rm EIP}(\rho,\,S) = \kappa\,\frac{(\boldsymbol{\nabla} S)^2}{2\,m}\,\rho^2\ . \tag{2.1.19}$$

This potential depends on ρ and on $\boldsymbol{\nabla}\,S$. Inserting Eq. (2.1.18) in Eq. (2.1.9) we obtain for the real part of the nonlinearity the expression:

$$W(\rho,\,S) = \kappa\,\frac{m}{\rho}\left(\frac{\boldsymbol{j}}{1+\kappa\,\rho}\right)^2 + F([\rho])\ , \tag{2.1.20}$$

with

$$F([\rho]) = \frac{\delta}{\delta\,\rho}\,\widetilde{U}([\rho])\ . \tag{2.1.21}$$

We have obtained the following result:
Starting from the expression of the quantum current $\boldsymbol{j}$ *appearing in the continuity equation, it is possible to deduce a NLSE compatible with it. This NLSE generally contains a complex nonlinearity. Only its imaginary part is fixed while the real one required an additional constraints. If we require that the system obeying the EIP is canonical, we obtain for the real part the quantity* $W(\rho,\,S)$ *gives by Eq. (2.1.20) that is defined modulo a real functional* $F([\rho])$.
In Ref. [77] the iter of the stochastic quantization leads to a NLSE with the same expression of $\mathcal{W}$ but a different form for W [see Eq. (1.2.30)].

Returning again to the fundamental fields ψ and ψ^*, the expression of the potential (2.1.19) takes the form:

$$U_{\text{EIP}}([\psi],\,[\psi^*]) = -\kappa\frac{\hbar^2}{8\,m}(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)^2\ , \tag{2.1.22}$$

so that we can write the most general form of the Lagrangian of a system obeying the EIP as:

$$\begin{aligned}\mathcal{L} \;&=\; i\,\frac{\hbar}{2}\left(\psi^*\,\frac{\partial\,\psi}{\partial\,t} - \psi\,\frac{\partial\,\psi^*}{\partial\,t}\right) - \frac{\hbar^2}{2\,m}\,|\boldsymbol{\nabla}\psi|^2 + \kappa\,\frac{\hbar^2}{8\,m}(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)^2\\ &-\; \widetilde{U}([|\psi|^2]) - V\,\psi^*\,\psi\ ,\end{aligned} \tag{2.1.23}$$

where $\widetilde{U}([|\psi|^2])$ is a real arbitrary smooth functional which might depend on ρ and its spatial derivatives.
In this thesis, if not otherwise specified, the arbitrary potential $\widetilde{U}(\rho)$ is an analytic nonderivative functional of ρ only. This potential can be used to describe other interactions in the system. By an appropriately choice of its form, the Lagrangian (2.1.1) can be used to describe different physical systems in presence of collective interactions and obeying to the EIP (see chapter V).

Using Eq.(2.1.3) we obtain the following NLSE:

$$\begin{aligned}i\,\hbar\,\frac{\partial\,\psi}{\partial\,t} \;=\;& -\;\frac{\hbar^2}{2\,m}\Delta\,\psi - \kappa\,\frac{\hbar^2}{2\,m}(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)\,\nabla\psi\\ & -\;\kappa\,\frac{\hbar^2}{4\,m}\,\boldsymbol{\nabla}(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)\,\psi + F(\rho)\,\psi + V\,\psi\ ,\end{aligned} \tag{2.1.24}$$

where $F(\rho)$ is now given by $F(\rho) = \partial\,\widetilde{U}(\rho)/\partial\,\rho$.
Eq. (2.1.24) can also be written using the expression of the current (2.1.14) and the particle density ρ in the form:

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t} = -\frac{\hbar^2}{2\,m}\Delta\,\psi + \Lambda(\rho,\,\boldsymbol{j})\,\psi + F(\rho)\,\psi + V\,\psi\ , \tag{2.1.25}$$

where the complex nonlinearity $\Lambda(\rho,\,\boldsymbol{j})$ is given by:

$$\Lambda(\rho,\,\boldsymbol{j}) = \kappa\,\frac{m}{\rho}\left(\frac{\boldsymbol{j}}{1+\kappa\,\rho}\right)^2 - i\,\kappa\,\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\left(\frac{\boldsymbol{j}\,\rho}{1+\kappa\,\rho}\right)\ . \tag{2.1.26}$$

The quantum system described by the Lagrangian density (2.1.1) is canonical. This can be verified defining the fields π_{ψ} and π_{ψ^*}, canonically conjugated to the

field ψ and ψ^*, by means of the relations (1.3.17):

$$\pi_{\psi} = i\,\frac{\hbar}{2}\,\psi^* \ , \tag{2.1.27}$$

$$\pi_{\psi^*} = -i\,\frac{\hbar}{2}\,\psi \ . \tag{2.1.28}$$

It is well known that π_{ψ} and π_{ψ^*} are proportional to the fields ψ^* and ψ, so that, while in the Lagrangian formalism ψ and ψ^* are independent fields, in the Hamiltonian formalism they are canonically conjugated. Following Ref.[81], Eqs. (2.1.27) and (2.1.28) give rise to the primary constrains:

$$\xi_1 = \pi_{\psi} - i\,\frac{\hbar}{2}\,\psi^* \ , \tag{2.1.29}$$

$$\xi_2 = \pi_{\psi^*} + i\,\frac{\hbar}{2}\,\psi \ . \tag{2.1.30}$$

Performing the Legendre transformation (1.3.18), it is easy to see that the Hamiltonian density can be written as:

$$\mathcal{H} = \frac{\hbar^2}{2\,m}\,|\boldsymbol{\nabla}\psi|^2 - \kappa\,\frac{\hbar^2}{8\,m}(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)^2 + \widetilde{U}(\psi^*\,\psi) + V\,\psi^*\,\psi \ . \tag{2.1.31}$$

Let us introduce now the Poisson brackets between two functionals:

$$\begin{aligned} \{f(\boldsymbol{x}),\,g(\boldsymbol{y})\} \;&=\; \int\left[\frac{\delta\,f(\boldsymbol{x})}{\delta\,\psi(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\pi_{\psi}(\boldsymbol{z})} - \frac{\delta\,f(\boldsymbol{y})}{\delta\,\pi_{\psi}(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{x})}{\delta\,\psi(\boldsymbol{z})}\right]\,d^Dz \\ &+\; \int\left[\frac{\delta\,f(\boldsymbol{x})}{\delta\,\psi^*(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\pi_{\psi^*}(\boldsymbol{z})} - \frac{\delta\,f(\boldsymbol{y})}{\delta\,\pi_{\psi^*}(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{x})}{\delta\,\psi^*(\boldsymbol{z})}\right]\,d^Dz \ . \end{aligned} \tag{2.1.32}$$

The second class primary derivative (2.1.29) and (2.1.30) satisfy the relation:

$$\{\xi_1(t,\,\boldsymbol{x}),\,\xi_2(t,\,\boldsymbol{y})\} = -i\,\hbar\,\delta^{(D)}(\boldsymbol{x}-\boldsymbol{y}) \ , \tag{2.1.33}$$

and can be accommodated by the introduction of the Dirac brackets:

$$\begin{aligned} \{f(\boldsymbol{x}),\,g(\boldsymbol{y})\}_D \;&=\; \{f(\boldsymbol{x}),\,g(\boldsymbol{y})\} + \frac{i}{\hbar}\int\{f(\boldsymbol{x}),\,\xi_1(\boldsymbol{z})\}\,\{\xi_2(\boldsymbol{z}),\,g(\boldsymbol{y})\}\,d^Dz \\ &-\; \frac{i}{\hbar}\int\{f(\boldsymbol{x}),\,\xi_2(\boldsymbol{z})\}\,\{\xi_1(\boldsymbol{z}),\,g(\boldsymbol{y})\}\,d^Dz \ . \end{aligned} \tag{2.1.34}$$

Expression (2.1.32) can be simplified if one solves the derivative (2.1.29), (2.1.30) for π_{ψ} and π_{ψ^*} and treats f and g as functionals of ψ and ψ^* only. A straightforward calculation gives:

$$\{f(\boldsymbol{x}),\,g(\boldsymbol{y})\}_D = \;-\;\frac{i}{\hbar}\int\left[\frac{\delta\,f(\boldsymbol{x})}{\delta\,\psi(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\psi^*(\boldsymbol{z})} - \frac{\delta\,g(\boldsymbol{y})}{\delta\,\psi(\boldsymbol{z})}\frac{\delta\,f(\boldsymbol{x})}{\delta\,\psi^*(\boldsymbol{z})}\right]\,d^Dz \ . \tag{2.1.35}$$

Using this expression we can obtain the evolution equations for the fields ψ and ψ^* in the Poisson formalism:

$$\frac{\partial \psi}{\partial t} = \{\psi, H\} , \tag{2.1.36}$$

$$\frac{\partial \psi^*}{\partial t} = \{\psi^*, H\} , \tag{2.1.37}$$

which give the Hamiltonian equations:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\delta H}{\delta \psi^*} , \tag{2.1.38}$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = \frac{\delta H}{\delta \psi} , \tag{2.1.39}$$

that are the Schrödinger equations for the fields ψ and ψ^*, respectively. It can be easily verified that the chose equations are equal to Eq. (1.3.13) introduced in section 1.3 with the Hamiltonian operator given by (1.3.20).

2.2 Hydrodynamic formulation

In this section we describe the NLSE with EIP in the hydrodynamic representation. Its utility will be seen in chapter V, where we study explicit solutions of the model. In particular we make use of the hydrodynamic formulation in order to obtain the solitary wave solutions of the system.

It is well known [83] that a quantum system can be seen as a Madelung-like fluid [84] described by means of the fields ρ and S trough a coupled system of differential equations. The velocity field associated to the fluid is related to the phase of ψ from the relation $\boldsymbol{v} = \boldsymbol{\nabla}\, S/m$ and its dynamics is described by means of a *Hamilton-Jacobi* like equation in presence of a nonlinear term function of ρ which was called quantum potential. Let us remember now that the dynamical equation for the phase is only formally similar to the Hamilton-Jacobi equation. In fact, in the classical mechanics, the Hamilton-Jacobi equation describes the evolution of the Hamilton principal function S that is the generator of a canonical transformation and is identified *a posteriori* with the action of the system: $S = \int L\, dt$. On the other hand, differently from the Bohm picture, the phase S of the field ψ is a dynamical field canonically conjugate to the density ρ and is not a generator of canonical transformations.

By introducing Eq. (2.1.4) in Eq. (2.1.24) and separating the real part from the

imaginary one, we obtain a coupled system in ρ and S:

$$\frac{\partial S}{\partial t}+\frac{(\boldsymbol{\nabla} S)^2}{2\,m}+\kappa\,\rho\,\frac{(\boldsymbol{\nabla} S)^2}{m}-\frac{\hbar^2}{2\,m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}+F(\rho)+V=0\ , \tag{2.2.1}$$

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\left[\frac{\boldsymbol{\nabla} S}{m}\,\rho\,(1+\kappa\,\rho)\right]=0\ . \tag{2.2.2}$$

Equation (2.2.1) is a *Hamilton-Jacobi* type equation, where the fourth term is the quantum potential [83]. The third term is the real part $W(\rho,\,\boldsymbol{j})$ of the term introduced by the EIP potential (2.1.19), as can be verified taking in mind the expression of the current $\boldsymbol{j}$ given by Eq. (2.1.14), that in the new field becomes:

$$\boldsymbol{j}=\frac{\boldsymbol{\nabla} S}{m}\,\rho\,(1+\kappa\,\rho)\ . \tag{2.2.3}$$

Finally the last two terms in Eq. (2.2.1) are the extra nonlinearity and the external potential. We can see that the imaginary part $\mathcal{W}(\rho,\,\boldsymbol{j})$ of the term introduced by the EIP potential does not appear in the *Hamilton-Jacobi* equation (2.2.1).
It is easy to recognize that Eq. (2.2.2) is the continuity equation in the ρ-S representation.

The *Hamilton-Jacobi* equation (2.2.1) and the continuity equation (2.2.2) can be derived from a variational principle, making use of the Euler operator (1.3.6) written in the ρ and S representation, obtained from Eq. (1.3.6) with the substitution $\psi\to\rho$ and $\psi\to S$ respectively. In fact, starting from the Lagrangian density $\widetilde{\mathcal{L}}$ which is given by:

$$\widetilde{\mathcal{L}}=-\frac{\partial S}{\partial t}\,\rho-\frac{(\boldsymbol{\nabla} S)^2}{2\,m}\rho-\kappa\,\frac{(\boldsymbol{\nabla} S)^2}{2\,m}\rho^2-\frac{\hbar^2}{8\,m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho}-\widetilde{U}(\rho)-V\,\rho\ , \tag{2.2.4}$$

and applied to the action $\widetilde{\mathcal{A}}[\rho,\,S]$ defined by (1.3.2) the Euler operator:

$$\mathrm{E}_\rho(\widetilde{\mathcal{A}})=0\ ,\qquad \mathrm{E}_S(\widetilde{\mathcal{A}})=0\ , \tag{2.2.5}$$

we obtain the Eqs. (2.2.1) and (2.2.2). In the Lagrangian density (2.2.4) the first three terms are the same as occurring in the linear Schrödinger equation, while the nonlinear contribution is given by the forth and six the term, where the fourth is the potential introduced by the EIP. The term $-\hbar^2\,(\boldsymbol{\nabla}\,\rho)^2/(8\,m\,\rho)$ in Eq. (2.2.4) is responsible of the presence of the quantum potential in Eq. (2.2.1), $U_q=-\hbar^2\,\Delta\rho^{1/2}/(2\,m\,\rho^{1/2})$. Notwithstanding, in literature the quantum potential is referred to the nonlinear term that appears in the *Hamilton-Jacobi* equation.

Now we introduce the Hamiltonian procedure. Let us show the results without describing the Dirac procedure.
The momentum π_S, canonically conjugate to S, is given by Eq. (1.3.17), that now becomes:

$$\pi_S = -\rho \; . \tag{2.2.6}$$

Moreover, we have $\pi_\rho = i\,\hbar/2$. Therefore, π_S is proportional to ρ, while π_ρ is a constant; the number of degrees of freedom is the same in both the Lagrangian and Hamiltonian formalism.
The Hamiltonian density, function of the canonically conjugate fields S and $-\rho$, can be deduced taking into account Eq. (1.3.18) and (2.2.4):

$$\tilde{\mathcal{H}} \;=\; \frac{(\boldsymbol{\nabla} S)^2}{2\,m}\,\rho+\kappa\,\frac{(\boldsymbol{\nabla} S)^2}{2\,m}\,\rho^2+\frac{\hbar^2}{8\,m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho}+\widetilde{U}(\rho)+V\,\rho \; . \tag{2.2.7}$$

The *Hamilton-Jacobi* and the continuity equations take the form:

$$\frac{\partial\,S}{\partial\,t}=-\frac{\delta\,\tilde{H}}{\delta\,\rho} \; , \tag{2.2.8}$$

$$\frac{\partial\,\rho}{\partial\,t}=\frac{\delta\,\tilde{H}}{\delta\,S} \; . \tag{2.2.9}$$

The same equations, in the Poisson formalism, can be rewritten as:

$$\frac{\partial\,S}{\partial\,t}=\{S,\,\tilde{H}\} \; , \tag{2.2.10}$$

$$\frac{\partial\,\rho}{\partial\,t}=\{\rho,\,\tilde{H}\} \; . \tag{2.2.11}$$

The evolution equations (2.1.38), (2.1.39) or (2.1.36), (2.1.37) deduced from $H(\psi,\,\psi^*)$ have the same form of the equations (2.2.8), (2.2.9) or (2.2.10), (2.2.11) deduced from $\tilde{H}(\rho,\,S)$. Then, according to a well-established procedure, we can relate the fields ψ-ψ^* to the fields S-ρ by means of a canonical transformation [80]. The equations of motion in the S-ρ representation will be used in chapter V to study particular soliton solutions of Eq. (2.1.25) that preserve their shapes in the time. As we will show in chapter V, we are able to decouple the system of equations (2.2.8), (2.2.9) or equivalently Eqs. (2.2.10), (2.2.11) obtaining a differential equation in the variable ρ only, whose solutions define the solitons of the systems with EIP.

In conclusion of this section we study the most simplest solutions of the NLSE (2.1.24) in presence of the nonlinearity introduced by EIP only, i.e. when we neglect the potentials involving $\widetilde{U}$ and V. These are the planar waves:

$$\psi(t,\,\mathbf{x})=A\,\exp\left[i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega\,t)\right] \; , \tag{2.2.12}$$

where $\boldsymbol{k}$ is the wave number and A a complex constant. After inserting Eq. (2.2.12) in (2.1.24) we obtain the following dispersion relation for the planar waves:

$$\omega = \frac{\hbar\,\boldsymbol{k}^2}{2\,m}\,(1 + 2\,\kappa\,|A|^2)\ , \tag{2.2.13}$$

which reproduces the well known dispersion relation of the Schrödinger equation when the EIP is switched off ($\kappa \to 0$). Let us remark that in presence of EIP, the angular frequency depends also on the amplitude of the wave. A this point it is important to remember that, differently from the linear theory, in the NLSE the normalization constant is not arbitrary. Generally, the amplitude of a solution is related to the other parameter of the system. For instance, the cubic NLSE [60] admits soliton solutions whose velocity is related to the amplitude. This amplitude increases when the soliton velocity decrease.
We remember also that because of the nonlinear nature of the systems the more general solution can not be expressible as superposition of planar waves how it is made in the linear case, by using the Fourier transformation method. More sophisticated mathematical tools, like, for instance, the inverse scattering method [64] are needed to found the general solutions. In this thesis we do not develop this procedure.
From Eq. (2.1.31), it appears that the system may be unstable in the case $\kappa < -1/|A|^2$. In fact, the energy of the modes with wave number $\boldsymbol{k}$ is:

$$E = \frac{\hbar^2\boldsymbol{k}^2}{2\,m}\,|A|^2\,(1 + \kappa\,|A|^2)\ , \tag{2.2.14}$$

which must be positive. This imposes the condition for the repulsive systems

$$|A|^2 \le \rho_{\rm max} \equiv 1/|\kappa| \tag{2.2.15}$$

according to the exclusion principle in the configuration space [77]

2.3 Physical observable

Now we study the time evolution of the average of the most important physical observable that describe the system in presence of EIP. We will identify the motion constants of the system. The results of this section will be obtained again in the next chapter from the analysis of the symmetries satisfied by the system. The section is organized in to parts: in the first we describe the general results. The mathematical proofs are collected in the second part.

2.3.1 Ehrenfest relations

Let us assume that the nonlinear potential $\widetilde{U}(\rho)$ and the field ψ vanish at infinity so that the surface terms can be disregarded. Moreover we assume that the potential $\widetilde{U}(\rho)$ depends on the space and on the time only through the field ρ. In this section we assume $D = 3$.
To obtain the Ehrenfest relations of the system obeying to Eq. (2.1.25) we start, at first, with the definition of average of an Hermitian operator $\widehat{A} = \widehat{A}^{\dagger}$:

$$< A >= \int \psi^* \, \widehat{A} \, \psi \, d^3x \ . \tag{2.3.1}$$

Hereinafter we normalize the system as:

$$N = \int \psi^* \, \psi \, d^3x \ , \tag{2.3.2}$$

where N is an integer, so that the field ρ assumes the meaning of a density of probability of position of a N-body system. Of course the normalization is maintained in the time thanks to the continuity equation (2.1.15):

$$\frac{d\,N}{d\,t} = 0 \ . \tag{2.3.3}$$

This means that the number of particles of the system do not change during the evolution. Using Eq. (1.3.21) it is easy to obtain the following relationship for the time evolution of the average of $\widehat{A}$:

$$\frac{d}{d\,t} < \widehat{A} >= \frac{i}{\hbar} \int \left[\frac{\delta H}{\delta \psi} \, \widehat{A} \, \psi - \psi^* \, \widehat{A} \, \frac{\delta H}{\delta \psi^*} \right] d^3x + < \frac{\partial \, \widehat{A}}{\partial \, t} > \ . \tag{2.3.4}$$

Here the last term takes eventually into account explicit time dependence of the functional A. Let us call $\widehat{\mathcal{O}}$ the operator in the r.h.s. of the NLSE (2.1.25) that can be rewritten in the form:

$$i\,\hbar \frac{\partial \, \psi}{\partial \, t} = \widehat{\mathcal{O}} \, \psi \ , \tag{2.3.5}$$

with

$$\widehat{\mathcal{O}} = \widehat{H_0} + W(\rho, \, \boldsymbol{j}) + i \, \mathcal{W}(\rho, \, \boldsymbol{j}) + F(\rho) \ , \tag{2.3.6}$$

where $\widehat{H_0} = (-\hbar^2/2\,m)\,\Delta + V(\boldsymbol{x})$ is the Hamiltonian operator of the linear theory. We can write the relation (2.3.4) in the following form:

$$\frac{d}{d\,t} < \widehat{A} >= \frac{i}{\hbar} < [\mathrm{Re}\, \widehat{\mathcal{O}}, \, \widehat{A}] > + \frac{1}{\hbar} < \{\mathrm{Im}\, \widehat{\mathcal{O}}, \, \widehat{A}\} > + < \frac{\partial \, \widehat{A}}{\partial \, t} > , \tag{2.3.7}$$

where the symbols $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ stand for the commutator and the anticommutator, respectively (see also Ref. [53] where an example of the Ehrenfest relations in a nonlinear Schrödinger equation with complex potential is discussed).

After setting $\widehat{A}=\widehat{\boldsymbol{x}}_c$ in Eq. (2.3.7), where $\widehat{\boldsymbol{x}}_c=\widehat{\boldsymbol{x}}/N$, we obtain the Ehrenfest relationship for the time evolution of the center of mass frame:

$$\frac{d}{d\,t} < \widehat{\boldsymbol{x}}_c >= -i\,\frac{\hbar}{2\,m\,N}\int(1+\kappa\,\rho)\,\left(\psi^*\,\boldsymbol{\nabla}\psi-\psi\,\boldsymbol{\nabla}\psi^*\right)\,d^3x\ . \tag{2.3.8}$$

It is worth remarking that the EIP introduces the additional quantity $<\kappa\,\rho\,\widehat{\boldsymbol{P}}>$ which is equal to $\kappa\,m\,\rho\,\boldsymbol{j}/(1+\kappa\,\rho)$. Note also that the right hand side of Eq. (2.3.8) can be written as:

$$\frac{d}{d\,t} < \widehat{\boldsymbol{x}}_c >= \frac{1}{N}\int\boldsymbol{j}\,d^3x\ , \tag{2.3.9}$$

which appear formally the same as in the standard linear quantum mechanics.

A second relation is obtained setting $\widehat{A}=\widehat{\boldsymbol{P}}$:

$$\frac{d}{d\,t} < \widehat{\boldsymbol{P}} >= -\int\psi^*\,\boldsymbol{\nabla}\,V\,\psi\,d^3x-\int\psi^*\,\boldsymbol{\nabla}\,F\,\psi\,d^3x\ . \tag{2.3.10}$$

Equation (2.3.10) can be regarded as the second law of the dynamics [85, 86]. The dynamics of the mean value of the momentum is governed by an *effective potential* given by the sum of the external potential V and of the nonlinearity $F(\rho)$. The EIP potential does not affect, on the average, the dynamics of the system because, due to their particular form, the terms W and $\mathcal{W}$ satisfy the relation $<[W,\nabla]-i\,\{\mathcal{W},\nabla\}>=0$. For the most frequent nonlinearity $F(\rho)$ generally appearing in the nonlinear Schrödinger equations the last term can be dropped and the Newtonian behavior is restored [85] (see the next section for the proof of this statement). On the contrary, other dynamical equations, like the sine-Gordon equation, seem to show a different behavior with respect to the Newtonian one. In this case the reason is in the kink-like solution of this equation that do not vanish on the boundary at infinity so that the surface terms can not be neglected.

Than, setting $\widehat{A}=\widehat{\boldsymbol{L}}$, where $\widehat{\boldsymbol{L}}$ is the angular momentum operator whose components are $\widehat{L}_i=\varepsilon_{ijk}\,x_j\,\widehat{P}_k$, we obtain:

$$\frac{d}{d\,t} < \widehat{\boldsymbol{L}} >= -\int\psi^*\,(\boldsymbol{x}\wedge\boldsymbol{\nabla}\,V)\,\psi\,d^3x-\int\psi^*(\boldsymbol{x}\wedge\boldsymbol{\nabla}\,F)\,\psi\,d^3x\ . \tag{2.3.11}$$

Like in the previous relation, the EIP potential does not contribute to the average of the angular momentum. Again if the nonlinear potential has a well behavior at infinity the last term is unremarkable on the dynamics of the system.

We discuss now the Ehrenfest relation concerning the energy. The energy of a canonical system, as we are considering here, is given by $E = H$ where H is the Hamiltonian given by Eq. (2.1.31). We can define a Hamiltonian operator $\widehat{H}$ whose average value is $<\widehat{H}>= H$. It is easily verified that:

$$\widehat{H} = -\frac{\hbar^2}{2\,m}\,\Delta + \frac{1}{\rho}\,U_{\text{EIP}}(\rho,\,\boldsymbol{j}) + \frac{1}{\rho}\,\widetilde{U}(\rho) + V(\boldsymbol{x},\,t)\ . \tag{2.3.12}$$

If we compare this expression of $\widehat{H}$ with the expression of the operator $\widehat{\mathcal{O}}$ given by Eq. (2.3.6) we find:

$$\widehat{H} \neq \widehat{\mathcal{O}}\ , \tag{2.3.13}$$

which means that the Hamiltonian operator of a nonlinear canonical system does not coincide with the operator $\widehat{\mathcal{O}}$ of the r.h.s. of the NLSE whilst, in the case of the linear theories, we have $\widehat{\mathcal{O}} = \widehat{H} = \widehat{H}_0$.
Within the definition $E =<\widehat{H}>$, we obtain the following relationship:

$$\frac{d\,E}{d\,t} =< \frac{\partial\,V}{\partial\,t} >\ , \tag{2.3.14}$$

which means that when time dependent external potentials are absent, the system is conservative being $d\,E/d\,t = 0$. We may conclude that the EIP does not introduce dissipative effects.
We remark that for a noncanonical model, where no Hamiltonian is present, the energy of the system is assumed generally as $E =<\widehat{\mathcal{O}}>$ where $\widehat{\mathcal{O}}$ is the operator of the r.h.s. of the corresponding NLSE. Several models with nonlinearities in the r.h.s. of the Schrödinger equation, characterized by time independent average values, have been developed. For instance, in the Kostin NLSE [87, 88], the operator $\widehat{\mathcal{O}}$ is defined as $\widehat{\mathcal{O}} = \widehat{H}_0 + (\hbar\,\gamma/2\,i)\,[\log(\psi/\psi^*) - <\log(\psi/\psi^*)>]$ being a real quantity, the energy of the system is defined as $E =<\widehat{\mathcal{O}}>$. In this case the non conservation of $<\widehat{\mathcal{O}}>$ implies energy dissipation of the system.

In conclusion, we have shown that the EIP potential (2.1.22) describes a conservative system. For a free system ($V = 0$) and when the non-linear potential $U(\rho)$ has a good behavior at infinity, we are able to identify four constants of motion:

$$N = \int \rho\, d^3x\ , \tag{2.3.15}$$

$$<\widehat{\boldsymbol{P}}>= -i\,\frac{\hbar}{2}\int \left(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*\right)\,d^3x\ , \tag{2.3.16}$$

$$<\widehat{\boldsymbol{L}}>= -i\,\frac{\hbar}{2}\int \boldsymbol{x}\times\left(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*\right)\,d^3x\ , \tag{2.3.17}$$

$$E = \int \mathcal{H}\,d^3x\ , \tag{2.3.18}$$

representing respectively the energy, the momentum, the angular momentum and the number of particles, conserved in virtue of the continuity equation (2.1.15).

2.3.2 Mathematical proofs

We deduce here the Ehrenfest relations for the observable N, $<\widehat{\boldsymbol{x}}_c>$, $<\widehat{\boldsymbol{P}}>$, $<\widehat{\boldsymbol{L}}>$, E given from Eqs. (2.3.2), (2.3.8), (2.3.10), (2.3.11) and (2.3.14). In the following we make the hypothesis that the fields vanish steeply at infinity and that we can neglect surface terms.
The first relation, Eq. (2.3.2), is a trivial consequence of the continuity equation, while the (2.3.8), can be easily obtained from Eq. (2.1.15) in the following way:

$$\frac{d}{d\,t}<\boldsymbol{x}_c>=\frac{1}{N}\int\boldsymbol{x}\,\frac{d\,\rho}{d\,t}\,d^3x=-\frac{1}{N}\int\boldsymbol{x}\,\boldsymbol{\nabla}\cdot\boldsymbol{j}\,d^2x=\frac{1}{N}\int\boldsymbol{j}\,d^3x\ , \tag{2.3.19}$$

where we have performed an integration by parts in the last step.
Let us consider now Eq. (2.3.10) for the component i of the momentum. Using Eq. (2.3.4) for $<\widehat{P}_i>=i(\hbar/2)\int(\psi^*\,\partial_i\psi-\psi\,\partial_i\psi^*)\,d^3x$ and taking into account the following functional derivative:

$$\begin{aligned}\frac{\delta\,<\widehat{P}_i>}{\delta\,\psi^*}&=i\,\hbar\,\partial_i\psi\ , &(2.3.20)\\ \frac{\delta H}{\delta\,\psi^*}&=\frac{\hbar^2}{2\,m}\,\Delta\psi+\kappa\,\frac{\hbar^2}{2\,m}\left[\psi^*\,\boldsymbol{\nabla}\psi-\psi\,\boldsymbol{\nabla}\psi^*\right]\,\boldsymbol{\nabla}\psi\\ &+\kappa\,\frac{\hbar^2}{4\,m}\,\boldsymbol{\nabla}\left[\psi^*\,\boldsymbol{\nabla}\psi-\psi\,\boldsymbol{\nabla}\psi^*\right]\,\psi+F(\rho)\,\psi+V\,\psi\ , &(2.3.21)\end{aligned}$$

and their conjugate, we obtain:

$$\begin{aligned}\frac{d}{d\,t}<P_i>&=\int\Bigg\{-\frac{\hbar^2}{2\,m}\left[\partial_i\psi^*\,\Delta\psi+\partial_i\psi\,\Delta\psi^*\right]\\ &+\kappa\,\frac{\hbar^2}{2\,m}\left[\psi^*\,\boldsymbol{\nabla}\psi-\psi\,\boldsymbol{\nabla}\psi^*\right]\left[\boldsymbol{\nabla}\psi^*\,\partial_i\psi-\boldsymbol{\nabla}\psi\,\partial_i\psi^*\right]\\ &+\kappa\,\frac{\hbar^2}{4\,m}\,\boldsymbol{\nabla}\left[\psi^*\,\boldsymbol{\nabla}\psi-\psi\,\boldsymbol{\nabla}\psi^*\right]\left[\psi^*\,\partial_i\psi-\psi\,\partial_i\psi^*\right]\\ &-\left[F(\rho)+V\right]\,\partial_i\,\rho\Bigg\}\,d^3x\ . &(2.3.22)\end{aligned}$$

Integrating by parts twice the first term and one time the third in the right hand side of (2.3.22) we obtain:

$$\frac{d}{d\,t}<P_i>=\int\Bigg\{-\frac{\hbar^2}{2\,m}\partial_i(\Delta\psi^*\,\psi)-\kappa\,\frac{\hbar^2}{8\,m}\partial_i\left[\psi^*\,\boldsymbol{\nabla}\psi-\psi\,\boldsymbol{\nabla}\psi^*\right]^2$$

$$- \quad [F(\rho) + V]\ \partial_i\, \rho \Big\}\, d^3x \ , \tag{2.3.23}$$

and neglecting surface terms we are left with:

$$\frac{d}{d\,t} < P_i >= \int \rho\, \partial_i\, V\, d^2x + \int \rho\, \partial_i\, F(\rho)\, d^2x \ . \tag{2.3.24}$$

Taking into account the relation $F = d\widetilde{U}/d\rho$, the last integral in (2.3.24) can be written as:

$$\int \partial_i\, (\rho\, F - \widetilde{U})\, d^2x \ , \tag{2.3.25}$$

and therefore, if the potential and the field ρ has a well behavior at infinity, it can be ignored. Then Eq. (2.3.24) is equal to Eq. (2.3.10).
Note that in Eq. (2.3.10) it does not appears the contribution of the nonlinear potential $\widetilde{U}(\rho)$. This statement is true also for nonlinear potential dependent on ρ and its spatial derivative $\widetilde{U}([\rho])$. In fact, in this case we have $F = \delta\, \widetilde{U}/\delta\, \rho$ and the last term in (2.3.24) is, as before, a surface integral.
For Eq. (2.3.11) we can follow the same procedure used for the relationship (2.3.10). We only give the trace of the procedure.
Setting $< \widehat{L}_i >= i(\hbar/2) \int \epsilon_{ijk}\, x_j\, (\psi^*\, \partial_k \psi - \psi\, \partial_k \psi^*)\, d^3x$, by using Eq. (2.3.4) we have:

$$\begin{aligned} \frac{d}{d\,t} < \widehat{L}_i > \; &= \; \int \epsilon_{ijk}\, x_j \Big\{ -\frac{\hbar^2}{2\,m} \left[\partial_k \psi^*\, \Delta\psi + \partial_k \psi\, \Delta\psi^*\right] \\ &+ \; \kappa\, \frac{\hbar^2}{2\,m} \left[\psi^*\, \boldsymbol{\nabla}\psi - \psi\, \boldsymbol{\nabla}\psi^*\right] \left[\boldsymbol{\nabla}\psi^*\, \partial_k \psi - \boldsymbol{\nabla}\psi\, \partial_k \psi^*\right] \\ &+ \; \kappa\, \frac{\hbar^2}{4\,m}\, \boldsymbol{\nabla} \left[\psi^*\, \boldsymbol{\nabla}\psi - \psi\, \boldsymbol{\nabla}\psi^*\right] \left[\psi^*\, \partial_k \psi - \psi\, \partial_k \psi^*\right] \\ &- \; [F(\rho) + V]\ \partial_k\, \rho \Big\}\, d^3x \ . \end{aligned} \tag{2.3.26}$$

Now we have perform the partial integration, as it was done in Eq. (2.3.22), taking into account that the quantity $\epsilon_{ijk} \partial_k\, x_j$ vanishes because the skew-symmetry of ϵ_{ijk} and disregarding the surface terms we obtain:

$$\frac{d}{d\,t} < \widehat{L}_i >= \int \epsilon_{ijk}\, x_j\, \partial_k\, [V + F(\rho)]\ \rho\, d^2x \ . \tag{2.3.27}$$

Computing as before the nonlinear quantity $F(\rho)$ we obtain Eq. (2.3.11).

Finally, we derive the relation (2.3.14). We begin writing the EIP potential as a function of the density $\rho = \psi^*\, \psi$ and of the gradient of the phase $S =$

$(i\,\hbar/2)\,\log(\psi^*/\psi)$. Equation (2.3.4) becomes:

$$\begin{aligned}\frac{d\,E}{d\,t} &= \int\left[-\frac{\hbar^2}{2\,m}\,\frac{\partial\,\psi^*}{\partial\,t}\,\Delta\,\psi-\frac{\hbar^2}{2\,m}\,\psi^*\,\Delta\,\frac{\partial\,\psi}{\partial\,t}\right.\\ &+ \left.\frac{\partial}{\partial\,t}\left(\widetilde{U}+U_{\text{EIP}}\right)+V\,\frac{\partial\,\rho}{\partial\,t}+\rho\,\frac{\partial\,V}{\partial\,t}\right]\,d^3x\ . \qquad (2.3.28)\end{aligned}$$

By using the equations of motion of the fields ψ and ψ^*:

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t} = \left[-\frac{\hbar^2}{2\,m}\,\Delta+V+F+W+i\,\mathcal{W}\right]\,\psi\ , \qquad (2.3.29)$$

$$-i\,\hbar\,\frac{\partial\,\psi^*}{\partial\,t} = \left[-\frac{\hbar^2}{2\,m}\,\Delta+V+F+W-i\,\mathcal{W}\right]\,\psi^*\ , \qquad (2.3.30)$$

with $\mathcal{W}$, W and F given by (2.1.16), (2.1.20) and (2.1.21) respectively, the Eq. (2.3.28) becomes:

$$\begin{aligned}\frac{d\,E}{d\,t} &= \int\left[i\,\frac{\hbar^3}{4\,m^2}\,\Delta\,\psi^*\,\Delta\,\psi-i\,\frac{\hbar}{2\,m}\,\psi^*\,\left(V+F+W-i\,\mathcal{W}\right)\,\Delta\,\psi\right]\,d^3x\\ &- \int\left[i\,\frac{\hbar^3}{4\,m^2}\,\psi^*\,\Delta^2\,\psi+i\,\frac{\hbar}{2\,m}\,\psi^*\,\Delta\,\left[\left(V+F+W+i\,\mathcal{W}\right)\,\psi\right]\right]\,d^3x\\ &+ \int\left[\left(V+\frac{\partial\,\widetilde{U}}{\partial\,\rho}+\frac{\partial\,U_{\text{EIP}}}{\partial\,\rho}\right)\,\frac{\partial\,\rho}{\partial\,t}+\frac{\partial\,U_{\text{EIP}}}{\partial(\boldsymbol{\nabla}\,S)}\,\frac{\partial\,(\boldsymbol{\nabla}\,S)}{\partial\,t}\right]\,d^3x\\ &+ <\frac{\partial\,V}{\partial\,t}>\ , \qquad (2.3.31)\end{aligned}$$

where we have used the relationship:

$$\frac{\partial}{\partial\,t}\,\left(\widetilde{U}+U_{\text{EIP}}\right)=\frac{\partial\,\widetilde{U}}{\partial\,\rho}\,\frac{\partial\,\rho}{\partial\,t}+\frac{\partial\,U_{\text{EIP}}}{\partial\,\rho}\,\frac{\partial\,\rho}{\partial\,t}+\frac{\partial\,U_{\text{EIP}}}{\partial(\boldsymbol{\nabla}\,S)}\,\frac{\partial\,(\boldsymbol{\nabla}\,S)}{\partial\,t}\ . \qquad (2.3.32)$$

In Eq. (2.3.31), integrating by parts and neglecting the surface terms, we obtain:

$$\begin{aligned}\frac{d\,E}{d\,t} &= -\frac{i\,\hbar}{2\,m}\int(V+F+W)\,\left(\psi^*\,\Delta\,\psi-\psi\,\Delta\,\psi^*\right)\,d^3x\\ &- \frac{\hbar}{2\,m}\int\mathcal{W}\,\left(\psi^*\,\Delta\,\psi+\psi\,\Delta\,\psi^*\right)\,d^3x\\ &+ \int\left\{\left(V+\frac{\partial\,\widetilde{U}}{\partial\,\rho}+\frac{\partial\,U_{\text{EIP}}}{\partial\,\rho}\right)\,\frac{\partial\,\rho}{\partial\,t}-\boldsymbol{\nabla}\left[\frac{\partial\,U_{\text{EIP}}}{\partial(\boldsymbol{\nabla}\,S)}\right]\,\frac{\partial\,S}{\partial\,t}\right\}\,d^3x\\ &+ <\frac{\partial\,V}{\partial\,t}>\ . \qquad (2.3.33)\end{aligned}$$

Using Eqs. (2.2.1) and (2.2.2), that we can rewrite in the form:

$$\frac{\partial\,\rho}{\partial\,t} = -\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla}\,S}{m}\,\rho\right) + \frac{2}{\hbar}\,\rho\,\mathcal{W}\ , \tag{2.3.34}$$

$$\frac{\partial\,S}{\partial\,t} = \frac{\hbar^2}{2\,m}\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}} - \frac{(\boldsymbol{\nabla}\,S)^2}{2\,m} - V - F - W\ , \tag{2.3.35}$$

and taking into account the relations:

$$\frac{\partial\,\widetilde{U}}{\partial\,\rho} + \frac{\partial\,U_{\text{EIP}}}{\partial\,\rho} = F + W \tag{2.3.36}$$

$$\boldsymbol{\nabla}\left[\frac{\partial\,U_{\text{EIP}}}{\partial(\boldsymbol{\nabla}\,S)}\right] = -\frac{2}{\hbar}\,\rho\,\mathcal{W}\ , \tag{2.3.37}$$

$$-\frac{i\,\hbar}{2\,m}\,\left(\psi^*\,\Delta\,\psi - \psi\,\Delta\,\psi^*\right) = \boldsymbol{\nabla}\,\left(\frac{\boldsymbol{\nabla}\,S}{m}\,\rho\right)\ , \tag{2.3.38}$$

$$\left(\psi^*\,\Delta\,\psi + \psi\,\Delta\,\psi^*\right) = 2\,\rho\,\left[\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}} - \left(\frac{\boldsymbol{\nabla}\,S}{\hbar}\right)^2\right]\ , \tag{2.3.39}$$

Eq. (2.3.33) becomes:

$$\begin{aligned}
\frac{d\,E}{d\,t} \;&=\; \int(V+F+W)\,\boldsymbol{\nabla}\,\left(\frac{\boldsymbol{\nabla}\,S}{m}\,\rho\right)\,d^3x - \frac{\hbar}{m}\int\rho\,\mathcal{W}\,\left[\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}} - \left(\frac{\boldsymbol{\nabla}\,S}{\hbar}\right)^2\right]\,d^3x\\
&+\; \int(V+F+W)\left[-\boldsymbol{\nabla}\left(\frac{\boldsymbol{\nabla}\,S}{m}\,\rho\right) + \frac{2}{\hbar}\,\rho\,\mathcal{W}\right]\,d^3x\\
&+\; \frac{2}{\hbar}\int\rho\,\mathcal{W}\,\left[\frac{\hbar^2}{2\,m}\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}} - \frac{(\boldsymbol{\nabla}\,S)^2}{2\,m} - V - F - W\right]\,d^3x\\
&+\; <\frac{\partial\,V}{\partial\,t}>\ .
\end{aligned} \tag{2.3.40}$$

We can immediately obtain:

$$\frac{d\,E}{d\,t} = <\frac{\partial\,V}{\partial\,t}>\ , \tag{2.3.41}$$

that is the relation (2.3.14). We note the absence of contribution of the nonlinear potentials $U_{\text{EIP}}(\rho,\,\boldsymbol{j})$ and $\widetilde{U}(\rho)$ to the average of the energy of the system. We conclude by seeing that relation (2.3.41) can be obtained straightforwardly by using the following property of the Poisson brackets:

$$\{f,\,f\} = 0\ , \tag{2.3.42}$$

valid for every functional f and therefore, we have:

$$\frac{d\,E}{d\,t} = \langle \frac{\partial\, V}{\partial\, t} \rangle \ , \tag{2.3.43}$$

if we take into account that explicit time dependence in the Hamiltonian can occur only in the external potential $V(t,\, \boldsymbol{x})$. Notwithstanding we have preferred to obtain it in a more rigorous fashion because of its importance.

Chapter III

Symmetries and Conservation Laws

The concept of symmetry plays an important role in the search of the solutions of dynamical equations. In the case of dynamical systems with infinite degrees of freedom described by partial differential equations (PDE), the Liouville integrability requires the knowledge of infinite symmetries. In fact many integrable PDE show this property like, for instance, the linear Schrödinger equation $i\,\psi_t + \psi_{xx} = 0$ [66], the Korteweg-de Vries (KdV) $u_t - 6\,u\,u_x + u_{xxx} = 0$ and the modified KdV (mKdV) $u_t - 6u^2\,u_x + u_{xxx} = 0$ [89], the cubic NLSE $i\,\psi_t + \psi_{xx} + |\psi|^2\,\psi = 0$ [90, 91], the Kaup-Newell equation $i\,\psi_t + \psi_{xx} - i(\psi\,\psi^*\,\psi)_x = 0$ [92], the Chen, Lee and Liu equation $i\,\psi_t + \psi_{xx} - i\,|\psi|^2\,\psi_x = 0$ [26] and others.
When the evolution equation can be obtained from a variational principle, the powerful Nöther theorem [82] gives the possibility to compute directly the associate conserved quantities satisfying appropriate continuity equations. The symmetries associated to geometrical transformations are very importants. In fact, as it is known, these are related to conserved physical quantities like the total energy, the linear and angular momentum of the system and so on.
In this chapter we study the symmetries described by the connected local Lie groups $\mathcal{G}$ depending on r parameters $\xi_i(t,\,\boldsymbol{x},\,\psi,\,\psi^*)$ which are functions of the independent variables t, $\boldsymbol{x}$ and the dependent ones ψ and ψ^*. They are continuous transformations mapping solutions of the evolution equations obeying the EIP in other solutions. We show that in presence of the potential introduced by EIP some geometrical symmetries of the linear Schrödinger equation are lost.
When the full group of symmetry $\mathcal{G}$ is known, we can study the set of solutions which are invariant under the action of symmetries belonging to the coset $\mathcal{G}/\mathcal{K}$ where $\mathcal{K}$ is the set of subgroup of $\mathcal{G}$ in $r-1$ parameters. In fact in this case, the original

PDE is reduced to an ordinary differential equation (ODE) which can be solved by quadrature. We perform this computation in chapter V where soliton solutions obeying the EIP are obtained with this procedure.
At the end of this chapter we will discuss in some detail a particular class of nonlinear gauged symmetries which can be considered as a generalization of the gauge transformation introduced recently by Doebner and Goldin [54]. Because it can be applied to a wide class of NLSE, we will study this transformation in a general contest. The importance of this transformation is that in order to make real the complex nonlinearity that occurs in the NLSE and as a consequence it linearizes the continuity equation. In some cases, known in the literature [54, 66, 93, 94, 95] it is also a powerful tool to linearize the NLSE at whole.

3.1 Lie Symmetries

Let us rewrite the EIP-Schrödinger equation (2.1.24), after an appropriate rescaling of the space-time coordinates, in the form:

$$i\,\psi_t + \psi_{\boldsymbol{xx}} + \kappa\,(\psi^*\,\psi_{\boldsymbol{x}} - \psi\,\psi^*_{\boldsymbol{x}})\,\psi_{\boldsymbol{x}} + \frac{\kappa}{2}\,(\psi^*\,\psi_{\boldsymbol{xx}} - \psi\,\psi^*_{\boldsymbol{xx}})\,\psi = 0 \ , \qquad (3.1.1)$$

where we use of the stenography notation: $\psi_{\boldsymbol{x}} \equiv \boldsymbol{\nabla}\,\psi$, $\psi_{\boldsymbol{xx}} \equiv \Delta\,\psi$.
We are concerned with the study of the symmetries of the EIP potential and thus, we neglect in this chapter both the nonlinear and the external potentials $\widetilde{U}(\rho)$ and V. The results obtained can be generalized when these quantities are present.
Being Eq. (3.1.1) canonical, we can perform the study of the symmetries both at the level of the Lagrangian and directly on the evolution equation. The study of the symmetries from the PDE is more general, because as we will show, not all the symmetries of the PDE are also symmetries of its Lagrangian. Therefore we start by studying the symmetries of Eq. (3.1.1) and after in the next section, starting from its Lagrangian, we derive, by means of Nöther theorem, the appropriate conserved quantities (when it occurs).
To find the Lie symmetries of Eq. (3.1.1) we adopt the geometric method presented in the Olver text-book [65] to which we send the reader for a complete treatment. We make a summary of the general concepts.
The first step in the research of the symmetries for a PDE is to look up it as an algebraic equation. For this purpose, we introduce the *jet*-space $\widetilde{F} \equiv M \times F \times F^{\boldsymbol{x}} \times F^{\boldsymbol{xx}} \times F^t$ which is an extension of the configuration space. Here M is the configuration space mapped by the space-time coordinates, F is the space mapped by the function ψ and ψ^* when are considered as independent fields. In the same fashion $F^{\boldsymbol{x}}$ is mapped from the function $\psi_{\boldsymbol{x}}$ and $\psi^*_{\boldsymbol{x}}$ and so on. In this

jet-space, Eq. (3.1.1) becomes an algebraic equation in the independent variables x, t, ψ, ψ^*, $\psi_{\boldsymbol{x}}$, $\psi^*_{\boldsymbol{x}}$, $\psi_{\boldsymbol{xx}}$, $\psi^*_{\boldsymbol{xx}}$ and ψ_t and defines an hyper-surface on $\widetilde{F}$. We consider now an element $g \in \mathcal{G}$ of a transformation Lie group which acts on the element of $\widetilde{F}$. We say that g is a symmetry transformation for Eq. (3.1.1) if a point P is mapped on the hyper-surface in a point on itself. In particular, if we consider infinitesimal transformations, we must require that those generated by the element g lie on the tangent plane on the hyper-surface in P.
Because $\mathcal{G}$ is a Lie group, its elements around the identity can be written as:

$$g = e^{i\,\epsilon\,\boldsymbol{v}} \;, \tag{3.1.2}$$

where ϵ is a parameter and $\boldsymbol{v}$ is a vector in the tangent plane on the hyper-surface. Its general expression will be:

$$\begin{aligned} \boldsymbol{v} &= \tau\,\partial_t + \xi^{\boldsymbol{x}} \cdot \partial_{\boldsymbol{x}} + \phi\,\partial_\psi + \widetilde{\phi}\,\partial_{\psi^*} + \phi^t\,\partial_{\psi_t} \\ &+ \phi^{\boldsymbol{x}} \cdot \partial_{\psi_{\boldsymbol{x}}} + \widetilde{\phi}^{\,\boldsymbol{x}} \cdot \partial_{\psi_{\boldsymbol{x}}} + \phi^{\,\boldsymbol{xx}} \cdot \partial_{\psi_{\boldsymbol{xx}}} + \widetilde{\phi}^{\,\boldsymbol{xx}} \cdot \partial_{\psi_{\boldsymbol{xx}}} \;, \end{aligned} \tag{3.1.3}$$

where the coefficients:

$$\xi^{\boldsymbol{x}} = \begin{pmatrix} \xi^x \\ \xi^y \\ \xi^z \end{pmatrix} \;, \qquad \phi^{\boldsymbol{x}} = \begin{pmatrix} \phi^x \\ \phi^y \\ \phi^z \end{pmatrix} \;, \qquad \widetilde{\phi}^{\,\boldsymbol{x}} = \begin{pmatrix} \widetilde{\phi}^{\,x} \\ \widetilde{\phi}^{\,y} \\ \widetilde{\phi}^{\,z} \end{pmatrix} \;,$$

$$\phi^{\boldsymbol{xx}} = \begin{pmatrix} \phi^{xx} \\ \phi^{yy} \\ \phi^{zz} \end{pmatrix} \;, \qquad \widetilde{\phi}^{\,\boldsymbol{xx}} = \begin{pmatrix} \widetilde{\phi}^{\,xx} \\ \widetilde{\phi}^{\,yy} \\ \widetilde{\phi}^{\,zz} \end{pmatrix} \;, \tag{3.1.4}$$

are three-vectors. All this quantities together with τ, ϕ and $\widetilde{\phi}$ are function of $\boldsymbol{x}$, t, ψ and ψ^*. The quantities $\phi^{\boldsymbol{x}}$, $\widetilde{\phi}^{\,\boldsymbol{x}}$, $\phi^{\boldsymbol{xx}}$, $\widetilde{\phi}^{\,\boldsymbol{xx}}$ and ϕ^t are related to the coefficients $\xi^{\boldsymbol{x}}$ and τ by means of:

$$\phi^i = \mathrm{D}_i\,\phi - (\mathrm{D}_i\,\xi^x)\;\psi_x - (\mathrm{D}_i\,\xi^y)\;\psi_y - (\mathrm{D}_i\,\xi^y)\;\psi_y - (\mathrm{D}_i\,\tau)\;\psi_t \;, \tag{3.1.5}$$

with $i \equiv x$, y, z, t where

$$\mathrm{D}_i \equiv \partial_i + \psi_i\,\partial_\psi + \psi^*_i\,\partial_{\psi^*} \;, \tag{3.1.6}$$

is the total derivative, and:

$$\phi^{ii} = \mathrm{D}_{ii}\,\phi - 2\;(\mathrm{D}_i\,\xi^x)\;\psi_{xx} - (\mathrm{D}_{ii}\,\xi^x)\;\psi_x + (x \to y) + (x \to z) + (x \to t) \;. \tag{3.1.7}$$

In Eqs. (3.1.5) and (3.1.7) the time derivative of the fields ψ and ψ^* can be eliminated by using the evolution equation (3.1.1). Of course, analogous expressions hold for $\widetilde{\phi}^{\,i}$ and $\widetilde{\phi}^{\,ii}$. At this point we must determine the coefficients τ, $\xi^{\boldsymbol{x}}$, ϕ and $\widetilde{\phi}$ so that the vector (3.1.3) lies on the tangent plane to the hyper-surface generated by Eq. (3.1.1)[3]. Let $L(\psi,\,\psi^*)$ denote the PDE:

$$L(\psi,\,\psi^*) \equiv i\,\psi_t + \psi_{\boldsymbol{xx}} + \kappa\,(\psi^*\,\psi_{\boldsymbol{x}} - \psi\,\psi^*_{\boldsymbol{x}})\,\psi_{\boldsymbol{x}} + \frac{\kappa}{2}\,(\psi^*\,\psi_{\boldsymbol{xx}} - \psi\,\psi^*_{\boldsymbol{xx}})\,\psi \ , \tag{3.1.8}$$

g is a symmetry transformation if:

$$L(g\,\psi,\,g\,\psi^*) = 0 \ , \tag{3.1.9}$$

whenever ψ and ψ^* are solutions of $L(\psi,\,\psi^*) = 0$. The condition for the vector $\boldsymbol{v}$ to lie on the tangent plane is given by [65]:

$$\boldsymbol{v}\left[i\,\psi_t + \psi_{\boldsymbol{xx}} + \kappa\,(\psi^*\,\psi_{\boldsymbol{x}} - \psi\,\psi^*_{\boldsymbol{x}})\,\psi_{\boldsymbol{x}} + \frac{\kappa}{2}\,(\psi^*\,\psi_{\boldsymbol{xx}} - \psi\,\psi^*_{\boldsymbol{xx}})\,\psi\right] = 0 \ . \tag{3.1.10}$$

Taking into account the expression of the vector $\boldsymbol{v}$ (3.1.3) we obtain:

$$\begin{aligned} & i\,\phi^t + \left(1 + \frac{\kappa}{2}\,\psi\,\psi^*\right)\,\phi^{\boldsymbol{xx}} - \frac{\kappa}{2}\,\psi^2\,\widetilde{\phi}^{\,\boldsymbol{xx}} + \kappa\,\left(2\,\psi^*\,\psi_{\boldsymbol{x}} - \psi\,\psi^*_{\boldsymbol{x}}\right)\,\phi^{\boldsymbol{x}} - \kappa\,\psi\,\psi_{\boldsymbol{x}}\,\widetilde{\phi}^{\,\boldsymbol{x}} \\ & + \frac{\kappa}{2}\,(\psi^*\,\psi_{\boldsymbol{xx}} - 2\,\psi\,\psi^*_{\boldsymbol{xx}} - 2\psi_{\boldsymbol{x}}\,\psi^*_{\boldsymbol{x}})\,\phi + \frac{\kappa}{2}\,\left(\psi\,\psi_{\boldsymbol{xx}} + 2\psi^2_{\boldsymbol{x}}\right)\,\widetilde{\phi} = 0 \ , \end{aligned} \tag{3.1.11}$$

which must be solved using the expression of the quantities ϕ, ϕ^i, ϕ^{ii}, $\tilde{\phi}^{\,i}$ and $\tilde{\phi}^{\,ii}$. By requiring that the coefficient of each monomial in ψ, ψ^* and their derivative vanishes separately, we obtain a sequence of derivative equations for the coefficients τ, $\xi^{\boldsymbol{x}}$, ϕ, $\widetilde{\phi}$ that when satisfied, give us the expression of the generators of the full Lie symmetries. The computation was developed with MATHEMATICA 2.0 package. We report in table the results:

$$\begin{array}{llll} \partial_{xx}\xi^x + \partial_{yy}\xi^x + \partial_{zz}\xi^x = 0 \ , & \partial_y\xi^x = -\partial_x\xi^y \ , & \partial_t\xi^t = 2\,\partial_x\xi^x \ , & \phi = \psi \ , \\ \partial_{xx}\xi^y + \partial_{yy}\xi^y + \partial_{zz}\xi^y = 0 \ , & \partial_z\xi^x = -\partial_x\xi^z \ , & \partial_x\xi^x = \partial_y\xi^y \ , & \widetilde{\phi} = -\psi^* \ , \\ \partial_{xx}\xi^z + \partial_{yy}\xi^z + \partial_{zz}\xi^z = 0 \ , & \partial_z\xi^y = -\partial_y\xi^z \ , & \partial_y\xi^y = \partial_z\xi^z \ , & \end{array} \tag{3.1.12}$$

and all the other derivatives are equal to zero. The general solution of this system can be easily obtained. Thus, we conclude that the most general infinitesimal symmetry

[3] In Ref. [65] the generator of the transformation $\boldsymbol{v}$ acting on the jet-space $\widetilde{F}$ was called prolongation of the vector $\boldsymbol{u}$ and denoted by $\boldsymbol{v} = pr^{(2)}\boldsymbol{u}$ where $\boldsymbol{u}$ is a vector on the tangent plane on M. Here we prefer to avoid this terminology for purposes of simplicity.

of the EIP-Schrödinger equation has coefficient functions of the form:

$$\xi^x = \lambda\,(y - z) + \mu\,x + \alpha^x \ , \tag{3.1.13}$$
$$\xi^y = \lambda\,(z - x) + \mu\,y + \alpha^y \ , \tag{3.1.14}$$
$$\xi^z = \lambda\,(x - y) + \mu\,z + \alpha^z \ , \tag{3.1.15}$$
$$\tau = 2\,\mu\,t + \alpha^t \ , \tag{3.1.16}$$
$$\phi = \beta\,\psi \ , \tag{3.1.17}$$
$$\widetilde{\phi} = -\beta\,\psi^* \ , \tag{3.1.18}$$

where λ, μ, β, α^i are real coefficients. Thus the Lie algebra of infinitesimal symmetries is spanned by the nine vector fields:

$$\left.\begin{array}{l} \boldsymbol{v}_1 = \partial_x \\ \boldsymbol{v}_2 = \partial_y \\ \boldsymbol{v}_3 = \partial_z \\ \boldsymbol{v}_4 = \partial_t \end{array}\right\} \qquad \text{translation,} \tag{3.1.19}$$

$$\left.\begin{array}{l} \boldsymbol{v}_5 = y\,\partial_x - x\,\partial_x \\ \boldsymbol{v}_6 = x\,\partial_z - z\,\partial_x \\ \boldsymbol{v}_7 = z\,\partial_y - y\,\partial_z \end{array}\right\} \qquad \text{rotational,} \tag{3.1.20}$$

$$\boldsymbol{v}_8 = t\,\partial_t + 2\,(x\,\partial_x + y\,\partial_y + z\,\partial_z) \qquad \text{dilation,} \tag{3.1.21}$$

$$\boldsymbol{v}_9 = \psi\,\partial_\psi - \psi^*\,\partial_{\psi^*} \qquad U(1)\ \text{global transformation.} \tag{3.1.22}$$

The groups $\mathcal{G}_i$ generated by the $\boldsymbol{v}_i$ are given in the following table. The entries give the transformed point $\exp(\epsilon\,\boldsymbol{v}_i)\,(t,\,\boldsymbol{x},\,\psi,\,\psi^*) = (\tilde{t},\,\tilde{\boldsymbol{x}},\,\tilde{\psi},\,\tilde{\psi^*})$:

$$\mathcal{G}_{1,2,3}: \qquad (t,\,\boldsymbol{x} + \boldsymbol{\epsilon},\,\psi,\,\psi^*) \ , \tag{3.1.23}$$
$$\mathcal{G}_4: \qquad (t + \epsilon,\,\boldsymbol{x},\,\psi,\,\psi^*) \ , \tag{3.1.24}$$
$$\mathcal{G}_{5,6,7}: \qquad (t,\,R^{ij}\,x_j,\,\psi,\,\psi^*) \ , \tag{3.1.25}$$
$$\mathcal{G}_8: \qquad (e^{2\,\epsilon}\,t,\,e^{\epsilon}\,\boldsymbol{x},\,\psi,\,\psi^*) \ , \tag{3.1.26}$$
$$\mathcal{G}_9: \qquad (t,\,\boldsymbol{x},\,e^{i\,\epsilon}\,\psi, e^{-i\,\epsilon}\,\psi^*) \ , \tag{3.1.27}$$

where R^{ij} is the 3×3 relational matrix. Since each group $\mathcal{G}_i$ is a symmetry group we have that if $\psi(t,\,\boldsymbol{x})$ is a solution of Eq. (3.1.1), so are the functions:

$$\psi^{(1)} = \psi(t,\,\boldsymbol{x} + \boldsymbol{\epsilon}) \ , \tag{3.1.28}$$
$$\psi^{(2)} = \psi(t + \epsilon,\,\boldsymbol{x}) \ , \tag{3.1.29}$$

$$\psi^{(3)} = \psi(t,\, R^{ij}\, x_j) \ , \tag{3.1.30}$$
$$\psi^{(4)} = \psi(e^{2\,\epsilon}\, t,\, e^{\epsilon}\, \boldsymbol{x}) \ , \tag{3.1.31}$$
$$\psi^{(5)} = e^{i\,\epsilon}\, \psi(t,\, \boldsymbol{x}) \ , \tag{3.1.32}$$

where ϵ and $\boldsymbol{\epsilon}$ are real quantities.

3.2 Conserved quantities

Having obtained the generators of the Lie symmetries and their algebrae, here we study the conserved quantities related to these symmetries, starting from the action of the system and by means of the Nöther theorem. In this section it is convenient to restore the standard unity of $\hbar$ and m. As it was said previously, not all the symmetries of the PDE (3.1.1) obtained in the last section are also symmetries for the action:

$$\mathcal{A} = \int \mathcal{L}\, d^3x\, dt \ . \tag{3.2.1}$$

As a consequence do not all those symmetries are related to conserved quantities. To start with, we remember that "*Lagrangian*" symmetries are those that do not change the formal expression of the action. Therefore, they are coordinates and fields transformations satisfying the relationship:

$$\delta\, \mathcal{A} = \int \Big[\delta\, \mathcal{L}\, d^3x\, dt + \mathcal{L}\, \delta(\, d^3x\, dt)\Big] = 0 \ . \tag{3.2.2}$$

The Nöther theorem states that a current $\mathcal{J}^\nu$ exists, given by:

$$\mathcal{J}^\nu = \frac{\partial\, \mathcal{L}}{\partial\, (\partial_\nu \psi)}\, \delta\, \psi + \frac{\partial\, \mathcal{L}}{\partial\, (\partial_\nu \psi^*)}\, \delta\, \psi^* - f^\nu \ , \tag{3.2.3}$$

satisfying the continuity equation:

$$\partial_\nu \mathcal{J}^\nu = 0 \ . \tag{3.2.4}$$

Therefore the quantity:

$$Q = \int\limits_{\mathcal{D}} \mathcal{J}^0\, d^2x \ , \tag{3.2.5}$$

is time conserved if the spatial component $\mathcal{J}^i$ vanishes steeply on the boundary $\partial\, \mathcal{D}$. Taking into account the result of the last section we begin considering the invariance of (3.1.1) under $U(1)$ transformations (3.1.22):

$$\psi'(t,\, \boldsymbol{x}) = e^{-i\alpha/\hbar}\, \psi(t,\, \boldsymbol{x}) \ . \tag{3.2.6}$$

The current is:

$$j^\mu \equiv \left(c\,\rho, -\frac{i\,\hbar}{2\,m}\,(1+\kappa\,\rho)\,(\psi^*\,\nabla\psi - \psi\,\nabla\psi)\right)\;, \tag{3.2.7}$$

and satisfies the continuity equation:

$$\partial_\mu\, j^\mu = 0\;. \tag{3.2.8}$$

The time component of Eq. (3.2.7) is the matter density ρ, the spatial part is the quantum current $\boldsymbol{j}$ with EIP.
Consider now the space-time translations with generator (3.1.19):

$$t \to t - t_0\;, \qquad \boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{x}_0\;, \tag{3.2.9}$$

the field $\psi(t,\,\boldsymbol{x})$ transforms as:

$$\psi(t,\,\boldsymbol{x}) \to \psi(t+t_0,\,\boldsymbol{x}+\boldsymbol{x}_0)\;. \tag{3.2.10}$$

These symmetries imply the conservation law:

$$\partial_\mu\, T^{\mu\,\nu} = 0\;, \tag{3.2.11}$$

where the energy-momentum tensor $T^{\mu\nu}$ obeys to EIP and is defined as:

$$T^{00} = \frac{\hbar^2}{2\,m}|\boldsymbol{\nabla}\,\psi|^2 - \kappa\,\frac{\hbar^2}{8m}\,(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)^2\;, \tag{3.2.12}$$

$$T^{0i} = -i\,c\,\frac{\hbar}{2}\,(\psi^*\,\partial_i\psi - \psi\,\partial_i\psi^*)\;, \tag{3.2.13}$$

$$\begin{aligned} T^{i0} = & -\frac{\hbar^2}{2\,m}\,(\partial_i\psi^*\,\partial_0\psi + \partial_i\psi\,\partial_0\psi^*) \\ & +\kappa\,\frac{\hbar^2}{4\,m}\,(\psi^*\,\partial_i\psi - \psi\,\partial_i\psi^*)\,(\psi^*\,\partial_0\psi - \psi\,\partial_0\psi^*)\;, \end{aligned} \tag{3.2.14}$$

$$\begin{aligned} T^{ij} = & \frac{\hbar^2}{2\,m}\,(\partial_i\psi^*\,\partial_j\psi + \partial_i\psi\,\partial_j\psi^*) \\ & -\kappa\,\frac{\hbar^2}{4\,m}\,(\psi^*\,\partial_i\psi - \psi\,\partial_i\psi^*)\,(\psi^*\,\partial_j\psi - \psi\,\partial_j\psi^*) \\ & -\delta_{ij}\left[\frac{\hbar^2}{4\,m}\,\Delta\rho + \kappa\,\frac{\hbar^2}{8m}\,(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\,\psi^*)^2\right]\;. \end{aligned} \tag{3.2.15}$$

Equations (3.2.12) and (3.2.13) define the generators of the transformations (3.2.9) and are the energy density and the momentum density of the system, respectively. From Eq. (3.2.12) we may define the quantity:

$$U_{\text{EIP}} = -\kappa\,\frac{\hbar^2}{8\,m}\,(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)^2\;, \tag{3.2.16}$$

representing the interaction energy density introduced by EIP. The quantity T^{00} is the sum of the kinetic energy density $(\hbar^2/2\,m)\,|\boldsymbol{\nabla}\psi|^2$ and the interaction energy density:

$$T^{00} = \frac{\hbar^2}{2\,m}\,|\boldsymbol{\nabla}\,\psi|^2 + U_{\rm EIP} \ . \tag{3.2.17}$$

From (3.2.13) we can see that the momentum density is not influenced by EIP. The energy flux T^{i0} and the momentum flux T^{ij} are modified by EIP, as we can see from Eqs. (3.2.14) and (3.2.15). Using Eq. (3.2.11) we remark that the quantities:

$$E = \int T^{00}\, d^3x \ , \tag{3.2.18}$$

$$P^i = \frac{1}{c}\int T^{0i}\, d^3x \ , \tag{3.2.19}$$

are constants of motion. By comparing Eqs. (3.2.13) and (3.2.14) we can see that the momentum density T^{0i} does not coincide with the energy flux density T^{i0} because of the non Lorentz invariance of the system. On the contrary we have $T^{ij} = T^{ji}$, suggesting spatial rotations invariance of the system.

In a nonrelativistic theory, like the one we are considering, the energy of the system must be a semidefinite positive quantity. This is a true statement for our model. In fact, taking into account the expression of the spatial component of (3.2.7), which represents the quantum current density, Eq. (3.2.12) of the energy density T^{00} becomes:

$$T^{00} = \frac{\hbar^2}{2\,m}|\boldsymbol{\nabla}\,\psi|^2 + \kappa\,\frac{m}{2}\left(\frac{\boldsymbol{j}}{1+\kappa\,\rho}\right)^2 \ . \tag{3.2.20}$$

This expression appears to be a semidefinite quantity when the parameter κ is positive. We can transform this expression in the form:

$$T^{00} = \frac{\hbar^2}{2\,m}|\boldsymbol{\nabla}\,\psi|^2\,(1+\kappa\,\rho) - \kappa\,\frac{\hbar^2}{8\,m}\,(\boldsymbol{\nabla}\,\rho)^2 \ , \tag{3.2.21}$$

which now results a semidefinite positive quantity when κ is negative, if we remember that the quantities $1+\kappa\,\rho$ is always positive.

Let us consider the transformations on the coordinate $\boldsymbol{x}$, produced by the orthogonal matrix R:

$$x^i \to R^i{}_j\, x^j \ , \qquad R^i{}_j\, R^j{}_k = \delta^i{}_k \ , \tag{3.2.22}$$

$$\psi(t,\,\boldsymbol{x}) \to \psi(t,\,R^{-1}\,\boldsymbol{x}) \ . \tag{3.2.23}$$

The action invariance allows us to define the angular momentum density:

$$M^{\mu ij} = x^j\, T^{\mu i} - x^i\, T^{\mu j}\ , \tag{3.2.24}$$

with $\mu = 0, \ldots, 3;\ \ i,\, j = 1,\, 2,\, 3$, obeying the continuity equation:

$$\partial_\mu M^{\mu ij} = 0\ . \tag{3.2.25}$$

We may note from Eqs. (3.2.13) and (3.2.15) that the generators of the transformation (3.2.23)

$$< L_i >= \varepsilon_{ijk} \int M^{0jk}\, d^3x\ , \tag{3.2.26}$$

are constants of motion, not affected by $U_{\rm EIP}$ which, on the contrary, modifies the flux densities.
We are left with the scaling symmetry generated by Eq. (3.1.21). It is easy to see that it is not a symmetry for the Lagrangian, independently of the number of the dimension D in which the system is immersed. Therefore we are not able to apply the Nöther theorem to obtain the correspondent conserved quantities.

We ask now what happens to the other generators of the Schrödinger group. In particular, from the list of symmetries obtained in section 3.1, the Galilei and the conformal symmetries are missed.

Let us consider the conformal group transformations:

$$t = \frac{\alpha\, t + \beta}{\gamma\, t + \delta}\ , \qquad \alpha\, \delta - \beta\, \gamma = 1\ , \tag{3.2.27}$$

with α, β, γ, δ arbitrary constants, that can be decomposed in three independent transformations: $t \to 1/t$, $t \to \alpha\, t$ and $1/t \to t + \delta$. The first is a discrete transformation and does not produce constants of motion, the other two, the dilation and the special conformal transformation, makes the action of the system not invariant. The reason why we loose the conformal symmetry lies in the fact that the parameter κ has a proper dimension. In order to analyze how the potential $U_{\rm EIP}$ breaks the symmetry, consider the case of dilation:

$$[\boldsymbol{x},\, t,\, \kappa,\, \psi(t, \boldsymbol{x})] \longrightarrow \left[\lambda\, \boldsymbol{x},\, \lambda^2\, t,\, \lambda^3\, \kappa,\, \lambda^{-3/2}\, \psi(t,\, \boldsymbol{x})\right]\ . \tag{3.2.28}$$

In this case, Eq. (3.2.2) becomes:

$$\delta\mathcal{A} = \int \left(\partial_\mu\, D^\mu + U_{\rm EIP}\right)\, d^3x\, dt = 0\ , \tag{3.2.29}$$

where:

$$D^\mu = 2\, t\, T^{\mu 0} - x_i\, T^{\mu i}\ , \tag{3.2.30}$$

is the well-known dilation current of Nöther. Therefore the potential U_{EIP} prevents to write the quantity $\partial_\mu D^\mu + U_{\text{EIP}}$ in the (3.2.28) as a tetradivergence. Note also that the conformal invariance is generally broken when derivative potentials [96], like U_{EIP}, are present. Nevertheless, if we consider the transformation:

$$[\boldsymbol{x},\, t,\, \kappa,\, \psi(t, \boldsymbol{x})] \longrightarrow \left[\lambda\, \boldsymbol{x},\, \lambda^2\, t,\, |\mu|^{-2}\, \kappa,\, \mu\, \psi(t,\, \boldsymbol{x})\right] \ , \tag{3.2.31}$$

with λ and μ arbitrary constants, Eq. (2.1.25) remains invariant. Remark that in Eq. (3.2.31) the transformation with parameter λ is the same of the scale transformation found in the last section while, the transformation with parameter μ is not a Lie symmetry. It permits us to reduce the study of the system described by Eq. (2.1.25) to consider only the two relevant cases with $|\kappa| = 1$.
We consider now the Galileo transformation on the coordinates:

$$t \to t \ , \qquad \boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{v}\, t \ , \tag{3.2.32}$$

and we set the field transformation ansatz:

$$\psi(t,\, \boldsymbol{x}) \to R(t,\, \boldsymbol{x})\, e^{i\,\alpha(t,\, \boldsymbol{x})}\, \psi(t,\, \boldsymbol{x} - \boldsymbol{v}\, t) \ , \tag{3.2.33}$$

with $\alpha(t,\, \boldsymbol{x})$ and $R(t,\, \boldsymbol{x})$ arbitrary real functions. The requirement that the action (3.2.1) be invariant under the transformations (3.2.32) and (3.2.33) gives the following derivative on the functions $\alpha(t,\, \boldsymbol{x})$ and $R(t,\, \boldsymbol{x})$:

$$\boldsymbol{\nabla}\alpha(t,\, \boldsymbol{x}) = -\frac{m\, \boldsymbol{v}}{\hbar}\frac{1}{1 + \kappa\, \rho} \ , \tag{3.2.34}$$

$$\frac{\partial\alpha(t,\, \boldsymbol{x})}{\partial t} = \frac{m\, \boldsymbol{v}^2}{2\, \hbar}\frac{1}{1 + \kappa\, \rho} \ , \tag{3.2.35}$$

$$R(t,\, \boldsymbol{x}) = const \ , \tag{3.2.36}$$

that, in the limit $\kappa \to 0$, reproduce the factor $\alpha(t,\, \boldsymbol{x}) = -m\, \boldsymbol{v} \cdot \boldsymbol{x} + m\, \boldsymbol{v}^2\, t/2$ of the linear theory. We can easily realize that in the general case $\kappa \neq 0$ Eqs. (3.2.34)-(3.2.35) are not satisfied by an arbitrary function of ρ.
The Galileo broken symmetry due to the potential (3.2.16) is not surprising. In fact we know that the generator of the Galilei transformation is given by:

$$\boldsymbol{G} = <\boldsymbol{P}>\ t - m\, N\ <\boldsymbol{x}_c> \ , \tag{3.2.37}$$

which represent the velocity center of mass of the system. If we derive with respect to time Eq. (3.2.37) and remembering that $<\boldsymbol{P}>$ is time independent because is a constant of motion, we obtain:

$$\frac{d\boldsymbol{G}}{dt} = <\boldsymbol{P}> - m\, N\, \frac{d <\boldsymbol{x}_c>}{dt} \ . \tag{3.2.38}$$

Taking into account the relations (2.3.9) and $< \boldsymbol{P} >= m \int \boldsymbol{j}_0 \, d^2x$ we have:

$$\frac{d\boldsymbol{G}}{dt} = -\kappa \, m \int \rho \, \boldsymbol{j}_0 \, d^2x \ , \tag{3.2.39}$$

where $\boldsymbol{j}_0$ is the current without EIP. We see immediately that the presence of EIP breaks the Galilei symmetry which is restored in the limit of $\kappa \to 0$.

We conclude with a brief discussion of the discrete transformations P and T. These are not broken by the presence of the potential (3.2.16), as it can be easily shown considering the following relations:

$$\begin{aligned} P: \qquad & (t,\, \boldsymbol{x}) \to (t,\, -\boldsymbol{x}) \ , \quad \psi(t,\, \boldsymbol{x}) \to \psi(t,\, -\boldsymbol{x}) \ , \\ & \boldsymbol{j}(t,\, \boldsymbol{x}) \to -\boldsymbol{j}(t,\, -\boldsymbol{x}) \ , \end{aligned} \tag{3.2.40}$$

$$\begin{aligned} T: \qquad & (t,\, \boldsymbol{x}) \to (-t,\, \boldsymbol{x}) \ , \quad \psi(t,\, \boldsymbol{x}) \to \psi^*(-t,\, \boldsymbol{x}) \ , \\ & \boldsymbol{j}(t,\, \boldsymbol{x}) \to -\boldsymbol{j}(-t,\, \boldsymbol{x}) \ . \end{aligned} \tag{3.2.41}$$

Therefore the expression of the potential (3.2.16) is invariant under the above transformations.

3.3 Nonlinear gauge transformations

In this section we describe a particular class of nonlinear gauge transformations allowing us to linearize the continuity equation making real the complex potential that appears in the evolution equation (2.1.25). Since the method that we are describing can be applied to a wide class of nonlinear Schrödinger equations, let us describe it in a general fashion and only at the end of the section we apply it to the EIP system. We introduce the method on the more simple 1-dimensional system, the generalization to the 3-dimensional case is in progress.

We introduce the density of Lagrangian

$$\mathcal{L} = i \, \frac{\hbar}{2} \left(\psi^* \, \frac{\partial \, \psi}{\partial \, t} - \psi \, \frac{\partial \, \psi^*}{\partial \, t} \right) - \frac{\hbar^2}{2 \, m} \left| \frac{\partial \, \psi}{\partial \, x} \right|^2 - U([\psi^*],\, [\psi]) \ , \tag{3.3.1}$$

which describes a class of one dimensional nonrelativistic and canonical quantum systems. In Eq. (3.3.1) the nonlinear real potential $U([\psi^*],\, [\psi])$ is a functional of the fields ψ and ψ^*. We will use the hydrodynamic fields $\rho(x,\, t)$ and $S(x,\, t)$ [cfr. Eqs. (2.1.5)-(2.1.6)]. The evolution equations of the fields $a \equiv \psi,\, \psi^*,\, \rho,\, S$ can be obtained from the action of the system $\mathcal{A} = \int \mathcal{L} \, dx \, dt$ by using the least action principle $E_a(\mathcal{A}) = 0$. We make use of the following notation for the spatial derivatives:

$$a_n = \frac{\partial^n \, a}{\partial \, x^n} \ , \qquad\qquad a_0 \equiv a \ . \tag{3.3.2}$$

It is immediate to obtain the evolution equation of the field ψ that is given by the following Schrödinger equation which contains a complex nonlinearity:

$$i\hbar\,\frac{\partial\,\psi}{\partial\,t} = -\frac{\hbar^2}{2\,m}\,\frac{\partial^2\,\psi}{\partial\,x^2} + W([\rho],\,[S])\,\psi + i\,\mathcal{W}([\rho],\,[S])\,\psi\ . \tag{3.3.3}$$

The real $W([\rho],\,[S])$ and the imaginary $\mathcal{W}([\rho],\,[S])$ part are given by the following expressions:

$$W([\rho],\,[S]) = \frac{\delta}{\delta\,\rho}\int U([\rho],\,[S])\,dx\,dt\ , \tag{3.3.4}$$

$$\mathcal{W}([\rho],\,[S]) = \frac{\hbar}{2\,\rho}\frac{\delta}{\delta\,S}\int U([\rho],\,[S])\,dx\,dt\ . \tag{3.3.5}$$

The evolution equation for the field ρ is obtained directly from $\mathrm{E}_a(\mathcal{A}) = 0$, posing $a \equiv S$ (now S is the field canonically conjugated to the field $-\rho$). We obtain the equation:

$$\frac{\partial\,\rho}{\partial\,t} + \frac{\partial\,j_{_\psi}}{\partial\,x} = \frac{\partial}{\partial\,S}\,U([\rho],\,[S])\ , \tag{3.3.6}$$

where the quantum current $j_{_\psi}$ takes the expression:

$$j_{_\psi} = \frac{S_1}{m}\,\rho + \sum_{n=0}(-1)^n\,\frac{\partial^n}{\partial\,x^n}\left[\frac{\partial}{\partial\,S_{n+1}}\,U([\rho],\,[S])\right]\ . \tag{3.3.7}$$

We remark that Eq. (3.3.6) is the continuity equation and the term in the right hand side represents a source for the field ρ. When the conservation of the number of particles $N = \int\rho\,dx$ is required, the hypothesis that the potential $U([\rho],\,[S])$ does not depend on S but only on its derivative must be introduced, therefore the Eq. (3.3.6) takes the form:

$$\frac{\partial\,\rho}{\partial\,t} + \frac{\partial\,j_{_\psi}}{\partial\,x} = 0\ . \tag{3.3.8}$$

We note that Eq. (3.3.8) can be obtained directly from (3.3.3) and from its complex conjugate, performing the standard procedure. We can see that the imaginary part $\mathcal{W}([\rho],\,[S])$ is responsible for the nonlinearity of the expression of the current $j_{_\psi}$ (3.3.7).
Let us introduce the following transformation for the field ψ:

$$\psi(x,\,t) \to \phi(x,\,t) = \mathcal{U}([\psi^*],\,[\psi])\,\psi(x,\,t)\ , \tag{3.3.9}$$

which allows us to eliminate the imaginary part of the evolution equation of the field ψ, which corresponds also to linearize the expression of the current $j_{_\psi}$.

The operator $\mathcal{U}$, generating this transformation, is unitary $\mathcal{U}^\dagger = \mathcal{U}^{-1}$ and is defined by:

$$\mathcal{U}([\psi^*],\,[\psi]) = \exp\left\{ i\,\frac{m}{\hbar}\sum_{n=0}(-1)^n\int\frac{1}{\rho}\frac{\partial^n}{\partial\,x^n}\left[\frac{\partial}{\partial\,S_{n+1}}\,U([\rho],\,[S])\right]\,dx\right\}\ . \quad (3.3.10)$$

If we write the field ϕ in terms of the hydrodynamic fields ρ, σ:

$$\phi(x,\,t) = \rho^{1/2}(x,\,t)\,\exp\left[\frac{i}{\hbar}\,\sigma(x,\,t)\right]\ , \quad (3.3.11)$$

and, due to the unitarity of the transformation, the modulo of ϕ is equal to the modulo of the field ψ, while the phase σ is given by:

$$\sigma = S + m\sum_{n=0}(-1)^n\int\frac{1}{\rho}\frac{\partial^n}{\partial\,x^n}\left[\frac{\partial}{\partial\,S_{n+1}}\,U([\rho],\,[S])\right]\,dx\ . \quad (3.3.12)$$

By accepting the statement made by Feynman and Hibbs ([97], p.96): "*Indeed all measurements of quantum-mechanical systems could be made to reduce eventually to position and time measurements*", (see also [54]) the two wave functions ψ and ϕ represent the same physical system and, as a consequence, we can interpret the Eq. (3.3.9) as a nonlinear gauge transformation of the function described by Eq. (3.3.3). From Eq. (3.3.3) and taking into account the transformation (3.3.9), it is easy to obtain the following evolution equation for the field ϕ:

$$\begin{aligned} i\,\hbar\,\frac{\partial\,\phi}{\partial\,t} &= -\frac{\hbar^2}{2\,m}\,\frac{\partial^2\,\phi}{\partial\,x^2} + W([\rho],\,[S])\,\phi \\ &- \frac{1}{2}\,m\left\{\sum_{n=0}(-1)^n\,\frac{1}{\rho}\,\frac{\partial^n}{\partial\,x^n}\left[\frac{\partial}{\partial\,S_{n+1}}\,U([\rho],\,[S])\right]\right\}^2\phi \\ &- m\sum_{n=0}(-1)^n\,\frac{\partial}{\partial\,t}\left\{\int\frac{1}{\rho}\,\frac{\partial^n}{\partial\,x^n}\left[\frac{\partial}{\partial\,S_{n+1}}\,U([\rho],\,[S])\right]\,dx\right\}\phi \\ &- \sum_{n=0}(-1)^n\,\frac{S_1}{\rho}\,\frac{\partial^n}{\partial\,x^n}\left[\frac{\partial}{\partial\,S_{n+1}}\,U([\rho],\,[S])\right]\phi\ . \end{aligned} \quad (3.3.13)$$

Note that the nonlinearity appearing in Eq. (3.3.13) is now real. The continuity equation of the system takes the form:

$$\frac{\partial\,\rho}{\partial\,t} + \frac{\partial\,j_{_\phi}}{\partial\,x} = 0 \quad (3.3.14)$$

where the current $j_{_\phi}$ has the standard expression of the linear quantum mechanics:

$$j_{_\phi} = \frac{\sigma_1}{m}\,\rho\ . \quad (3.3.15)$$

The gauge transformation (3.3.9) and (3.3.10) makes real the complex nonlinearity in the evolution equation, and makes non canonical the new dynamical system. However, this transformation may be useful to describe the evolution of system by means of an equation containing a real nonlinearity. We remark that nonlinear transformations have been introduced and used systematically for the first time in order to study nonlinear Schrödinger equations as the Doebner-Goldin one in Ref. [54].
The method here proposed can be found in literature applied to equations describing systems of collectively interacting particles.
We can quote for instance the canonical Doebner-Goldin equation [54, 95] that can be obtained from (3.3.1) when the potential $U([\rho],\,[S])$ has the following form:

$$U([\rho],\,[S])=\frac{D}{2}\,(\rho_1\,S_1-\rho\,S_2)\ . \tag{3.3.16}$$

A complex nonlinearity is generated in the evolution equation of the field ψ, with real and imaginary part given respectively by:

$$W([\rho],\,[S])=-m\,D\,\frac{\partial}{\partial\,x}\left(\frac{j_\psi}{\rho}\right)\ , \tag{3.3.17}$$

$$\mathcal{W}([\rho],\,[S])=\frac{\hbar\,D}{2\,\rho}\,\frac{\partial^2\,\rho}{\partial\,x^2}\ . \tag{3.3.18}$$

The quantum current j_ψ takes the form of a Fokker-Planck current:

$$j_\psi=\frac{S_1}{m}\,\rho+D\,\rho_1\ , \tag{3.3.19}$$

resulting to be the sum of two terms, the former is a drift current while the latter is a Fick current. The generator of the transformation $\mathcal{U}$ (3.3.10) takes the form:

$$\mathcal{U}([\psi^\ast],\,[\psi])=\exp\left(\frac{i}{\hbar}\,m\,D\,\log\rho\right)\ . \tag{3.3.20}$$

This is a particular case of a class of transformations introduced by Doebner and Goldin and permits to write the evolution equation:

$$i\,\hbar\,\frac{\partial\,\phi}{\partial\,t}=-\frac{\hbar^2}{2\,m}\,\frac{\partial^2\,\phi}{\partial\,x^2}+2\,m\,D^2\,\frac{1}{\rho^{1/2}}\,\frac{\partial^2\,\rho^{1/2}}{\partial\,x^2}\,\phi\ . \tag{3.3.21}$$

Equation (3.3.21) was studied in Ref. [98]; after rescaling: $\sigma\to\sqrt{1-(2\,m\,D/\hbar)^2}\,\sigma$, it reduces to the linear Schrödinger equation.

Now we describe the nonlinear gauge transformation applied to the Schrödinger equation with EIP. The system that we want transform is gives by Eq. (2.1.25) that we rewrite here in 1-dimensional space:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\Lambda(\rho,\,j_{\psi})\,\psi+F(\rho)\,\psi+V\,\psi\ , \tag{3.3.22}$$

with:

$$\Lambda(\rho,\,j_{\psi})=W(\rho,\,j_{\psi})+i\,\mathcal{W}(\rho,\,j_{\psi})\ , \tag{3.3.23}$$

and

$$W(\rho,\,j_{\psi}) = k\,\frac{m}{\rho}\left(\frac{j_{\psi}}{1+\kappa\,\rho}\right)^2\ , \tag{3.3.24}$$

$$\mathcal{W}(\rho,\,j_{\psi}) = -\kappa\,\frac{\hbar}{2\,\rho}\,\nabla\left(\frac{j_{\psi}\,\rho}{1+\kappa\,\rho}\right)\ . \tag{3.3.25}$$

Therefore, we introduce the unitary gauge transformation $\mathcal{U}$ for the field ψ:

$$\psi(x,\,t)\rightarrow\phi(x,\,t)=\mathcal{U}([\rho],\,[S])\,\psi(x,\,t)\ , \tag{3.3.26}$$

which acts on the phase of ψ as:

$$\frac{\partial S}{\partial x}\rightarrow\frac{\partial\sigma}{\partial x}=\frac{\partial S}{\partial x}\,(1+\kappa\,\rho)\ . \tag{3.3.27}$$

It is easy to see that $\mathcal{U}$ is given by:

$$\mathcal{U}([\rho],\,[S])=\exp\left(i\,\frac{\kappa}{\hbar}\int\rho\,\frac{\partial S}{\partial x}\,dx\right)\ . \tag{3.3.28}$$

The current j_{ϕ}, associated to the new field ψ, takes now the standard form of the linear quantum mechanics:

$$j_{\phi}=\frac{1}{m}\frac{\partial\sigma}{\partial x}\,\rho\ , \tag{3.3.29}$$

while the continuity equation is written as:

$$\frac{\partial\rho}{\partial t}+\frac{\partial j_{\phi}}{\partial x}=0\ . \tag{3.3.30}$$

The evolution equation for the field ϕ is again nonlinear:

$$i\hbar\frac{\partial\phi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2}+\widetilde{\Lambda}(\rho,\,j_{\phi})\,\phi+F(\rho)\,\phi+V\,\phi\ ,$$

with

$$\widetilde{\Lambda}(\rho,\, j_{\phi}) = \kappa\, m\, \frac{j_{\phi}^2}{\rho\,(1+\kappa\,\rho)}\, \phi - \kappa\, \frac{\hbar^2}{4\, m} \left[\frac{\partial^2 \rho}{\partial\, x^2} - \frac{1}{\rho} \left(\frac{\partial\, \rho}{\partial\, x} \right)^2 \right] \phi \;, \tag{3.3.31}$$

but now it is a real quantity. Note also that the transformation (3.3.26) does not affect the extra nonlinearity $F(\rho)$ because $|\psi|^2 = |\phi|^2 = \rho$. Of course the price that we pay to make real the quantity $\Lambda(\rho,\, j)$ given by (3.3.23) is that the new system, described by ϕ, is noncanonical because of the nonlinearity of the transformation.

Chapter IV

EIP-Gauged Schrödinger Model

In this chapter we introduce the gauged EIP-Schrödinger model describing collective effects in interacting particles systems with the Lagrangian (2.1.23) coupled in a minimal fashion with gauge fields A_μ that takes values in the abelian group $U(1)$. Many are the applications of NLSEs coupled with a gauge field and found in literature. One of the more important example is given by the Ginzburg-Landau theory of the superconductivity [99], where the same NLSE of Refs. [39, 100] with nonlinearity $\propto\ \rho$ is coupled with an Abelian gauge field, the interaction of this one being described by the Maxwell Lagrangian [47, 101, 102].

In the model studied by us, the interaction of gauge fields are described by the more complicated Maxwell-Chern-Simons Lagrangian (MCS). Many are the motivation to consider the MCS interaction respect to the more easy case when only the Maxwell term is present. It is well known that the Maxwell theory could be defined in any space-time dimension; the field strength tensor is still the antisymmetric one $F_{\mu\nu}$, the Maxwell Lagrangian $\mathcal{L} \propto F_{\mu\nu}\, F^{\mu\nu}$ and the equations of motion do not change their form. The only difference is in the number of independent fields contained in the theory. Differently, when the system is imbedded in a even space-time dimensions, there are two possible expressions for a first order derivative gauge Lagrangian which are both gauge and Lorentz invariant. The first one is the standard Maxwell Lagrangian, while the other, $\mathcal{L} \propto \epsilon^{\mu\nu\sigma\tau}\, F_{\mu\nu}\, F_{\sigma\tau}$, can be shown to be a pure divergence and therefore does not give contribution to the motion equations. The situation changes drastically when the dynamics of a physical system is developed in odd space-time dimensions. In this case the interaction of gauge fields can be described either by the Maxwell that also the Chern-Simons (CS) terms.
As a consequence, because of the presence of the CS term, the model is useful to describe physical systems with planar dynamics. At this time the reader might wonder if our discussion is merely of academic interest. The answer to the question

is negative. In fact, two dimensional physics can occur in our three-dimensional world. This is because of the third law of thermodynamics, which states that all the degrees of freedom are frozen out in the limit of zero temperature; it is possible to strictly confine the electrons to surfaces. Therefore it may happen that in a strongly confining potentials, or at sufficiently low temperatures, the excitation energy in one direction may be much higher than the average thermal energy of the particles, so that these dimensions are effectively frozen out.

How it was suggested by Wilczek [19] the presence of CS term confers to the system an anyonic behavior [28], that now obeys to a non conventional statistics [27]. Therefore field theory in presence of CS coupling can describe phenomenologies in which particles or elementary excitations could be anyons. This hope has in fact been realized in the case of the fractional Hall effect where the quasi-particles are believed to be charged vortices obeying to anyonic statistics [103]. Recent experiments [104] seem to confirm the existence of fractionally charged excitations and hence indirectly of anyons.
Another topic in condensed matter physics where CS term is believed to be correct, is the recently discovered high-T_c superconductors, characterized by their two-dimensional nature [20, 105]. This hypothesis on the usefulness of the CS term is confirmed by the P and T violating symmetries which are observed in this materials [106, 107, 108].
We remember also that CS term is an alternative method respect to the Proca Lagrangian for giving mass to the gauge field, without breaking the gauge invariance [109, 110, 111, 112, 113]. Moreover, it provides an example of topological field theory (for a review see [114]) since even in a curve space-time, the action of the CS term has the same form without any additional metric insertions. This fact has important consequences because, how we will show, the CS term does not give contribution to the energy-momentum tensor of the system.

4.1 MCS model with EIP

Let us introduce the density of Lagrangian of the Schrödinger equation obeying to an exclusion-inclusion principle with Maxwell-Chern-Simons interaction:

$$\mathcal{L} = \mathcal{L}_{\rm mat} + \mathcal{L}_{\rm gauge} \ , \tag{4.1.1}$$

where $\mathcal{L}_{\rm mat}$ is obtained from Eq. (2.1.23) after the substitution:

$$\partial_\mu \to D_\mu \equiv \partial_\mu + \frac{i\,e}{\hbar\,c}A_\mu \ , \tag{4.1.2}$$

and takes the final form:

$$\mathcal{L}_{\rm mat} = i\,c\,\frac{\hbar}{2}\,[\psi^*\,D_0\psi - \psi\,(D_0\psi)^*] - \frac{\hbar^2}{2\,m}\,|\boldsymbol{D}\psi|^2$$
$$+\kappa\,\frac{\hbar^2}{8\,m}[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*]^2 - \widetilde{U}(\psi^*\,\psi) - V\,\psi^*\,\psi\ , \tag{4.1.3}$$

where we have denoted the spatial component of the covariant derivative with $\boldsymbol{D} \equiv \boldsymbol{\nabla} - i\,(e/\hbar\,c)\,\boldsymbol{A}$, the indices are lower and upper depending on metric tensor in the Minkowski space $\eta_{\mu\nu} \equiv \mathrm{diag}(1,\,-1,\,-1)$: $A^\mu = \eta^{\mu\nu}A_\nu$; $\widetilde{U}(\rho)$ is an analytic real potential function of the field $\rho = |\psi|^2$, V is an external potential, m is the mass parameter and κ the coupling constant for the EIP potential. The quantity:

$$U_{\rm\scriptscriptstyle EIP} = -\kappa\,\frac{\hbar^2}{8\,m}\,[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*]^2\ \ , \tag{4.1.4}$$

is the EIP potential with minimal coupling [cfr. Eq. (2.1.22)].
Of course the model might be study in any spatial dimension D (if D is even the CS term is absent), but for the purposes of application that will be developed in the following chapters, we focus our attention in the planar case $D = 2$. Because the system is in (2+1) dimensions, the greek indices take the value 0, 1, 2 while the latin indices, that assume the value 1, 2, are the spatial ones. We have, $x_\mu \equiv (c\,t,\,-\boldsymbol{x})$, $\partial_\mu \equiv (c^{-1}\,\partial/\partial\,t,\,\boldsymbol{\nabla})$. In (4.1.2) c is the speed of light and e is the coupling constant of the Abelian gauge field described by the scalar potential A^0 and the vector potential $\boldsymbol{A}$, with $A_\mu \equiv (A_0,\,-\boldsymbol{A})$. Moreover, we assume the sum convention when the indices are repeated.

The Lagrangian $\mathcal{L}_{\rm gauge}$ is given by:

$$\mathcal{L}_{\rm gauge} = -\frac{\gamma}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{g}{4}\,\varepsilon^{\tau\mu\nu}\,A_\tau\,F_{\mu\nu}\ , \tag{4.1.5}$$

with $F_{\mu\nu} = \partial_\mu\,A_\nu - \partial_\nu\,A_\mu$. The Levi-Civita tensor $\varepsilon^{\tau\mu\nu}$, fully antisymmetric, is defined as $\varepsilon^{012} = 1$. The parameters γ and g in Eq. (4.1.5) give the relative weight between the Maxwell interaction and the CS one. The result obtained in this chapter, posing $g = 0$ or $\gamma = 0$ respectively, hold also when the Maxwell term or the Chern-Simons one are present alone. Because of the planarity of the system, the electric field is a vector with component:

$$E^i = -\partial_0\,A^i - \partial_i\,A^0\ , \tag{4.1.6}$$

while the magnetic field becomes a scalar:

$$B = -\epsilon^{ij}\,\partial_i\,A_j\ . \tag{4.1.7}$$

In terms of the component of the tensor $F^{\mu\nu}$ we have: $E^i = F_{0i},\ B = -F_{12}$.
Now we introduce the action of the system:

$$\mathcal{A} = \int \mathcal{L}\, d^2x\, dt \ . \tag{4.1.8}$$

The motion equations for the matter field ψ and for the gauge fields A_μ can be written as:

$$\mathrm{E}_{\psi^*}\mathcal{A} = 0 \ , \qquad\qquad \mathrm{E}_{A_\mu}\mathcal{A} = 0 \ , \tag{4.1.9}$$

where E_{A_μ} is obtained from (1.3.6) with the substitution $\psi \to A_\mu$. Explicitly, Eq. (4.1.9) becomes:

$$\begin{aligned} i\,\hbar\, c\, D_0\,\psi &= -\frac{\hbar^2}{2\,m}\,\boldsymbol{D}^2\psi - \kappa\,\frac{\hbar^2}{2\,m}\left[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*\right]\,\boldsymbol{D}\psi \\ &-\kappa\,\frac{\hbar^2}{4\,m}\,\boldsymbol{D}\left[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*\right]\,\psi + F(\rho)\,\psi + V\,\psi \ . \end{aligned} \tag{4.1.10}$$

If we introduce the spatial current:

$$\boldsymbol{J} = -\frac{i\,\hbar}{2\,m}\,(1+\kappa\,\rho)\,\left[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*\right] \ , \tag{4.1.11}$$

given by $\boldsymbol{J} = (1+\kappa\,\rho)\,\boldsymbol{J}_0$, being

$$\boldsymbol{J}_0 = -\frac{i\,\hbar}{2\,m}\left[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi^*)\right] = \boldsymbol{j}_0 - \frac{e}{m\,c}\,\boldsymbol{A}\,\rho \ , \tag{4.1.12}$$

the spatial current of the MCS theory without the EIP, it is easy to see that Eq. (4.1.10) obeys to the following continuity equation:

$$\frac{\partial\,\rho}{\partial\,t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \ . \tag{4.1.13}$$

Eq. (4.1.10) can be rewritten in terms of the fields ρ and $\boldsymbol{J}$ in the form:

$$i\,\hbar\,c\,D_0\,\psi = -\frac{\hbar^2}{2\,m}\boldsymbol{D}^2\psi + \Lambda(\rho,\,\boldsymbol{J})\,\psi + F(\rho)\,\psi + V\,\psi \ , \tag{4.1.14}$$

where the complex nonlinearity $\Lambda(\rho,\,\boldsymbol{J})$ is given by:

$$\Lambda(\rho,\,\boldsymbol{J}) = \kappa\,\frac{m}{\rho}\left(\frac{\boldsymbol{J}}{1+\kappa\,\rho}\right)^2 - i\,\kappa\,\frac{\hbar}{2\,\rho}\,\boldsymbol{D}\left(\frac{\boldsymbol{J}\,\rho}{1+\kappa\,\rho}\right) \ . \tag{4.1.15}$$

The motion equations for the fields A_μ, given from the second of Eqs. (4.1.9), are:

$$\gamma\,\partial_\mu\,F^{\mu\nu} + \frac{g}{2}\,\varepsilon^{\nu\tau\mu}\,F_{\tau\mu} = \frac{e}{c}\,J^\nu \ , \tag{4.1.16}$$

where the covariant current J^ν is given by: $J^\nu \equiv (c\,\rho,\,\boldsymbol{J})$.
Eq. (4.1.16) requires same comments. For $g = 0$ we recognize the standard Maxwell equations with sources. It is well known that this equation does not admit any trivial solution also in absence of the matter field $\boldsymbol{J}$. Differently, when we set $\gamma = 0$ the Chern-Simons equations for the gauge fields are obtained. This equations does not admit trivial solutions only in presence of matter. In fact how we will show in section 4.2, in absence of the Maxwell term, the gauge fields can be expressed as nonlinear functions of the field ψ, and vanish in absence of the matter field. Therefore, when the Maxwell term is present, the gauge fields describe dynamical fields with proper degrees of freedom, while in absence of this one, the CS term can be see as a constrain which the matter field must obey. Is this constrain that confer to the system an anyonic behavior.
Now,we perform the gauge transformations:

$$A_\mu \ \rightarrow \ A_\mu + \partial_\mu\omega \ , \tag{4.1.17}$$
$$\psi \ \rightarrow \ e^{-i(e/\hbar\,c)\,\omega}\psi \ , \tag{4.1.18}$$

where ω is a well-behaved function so that $\epsilon^{\mu\nu}\partial_\mu\,\partial_\nu\,\omega = 0$ (with $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$). The Lagrangian (4.1.1) changes as:

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{g}{4}\,\epsilon^{\mu\nu\tau}\,\partial_\mu\,(\omega\,F_{\nu\tau}) \ , \tag{4.1.19}$$

with an extra surface term that does not change the motion equations of the fields ψ and A_μ. This property of the system is typical in the presence of CS term and continues to be valid also when the EIP interaction is introduced. From Eq. (4.1.16), contracting with the differential operator ∂_ν, we may obtain again the conservation law of the current (4.1.13) in the covariant form:

$$\partial_\nu\,J^\nu = 0 \ . \tag{4.1.20}$$

The time component of Eq. (4.1.16) is the Gauss law:

$$\gamma\,\boldsymbol{\nabla}\cdot\boldsymbol{E} - g\,B = e\,\rho \ , \tag{4.1.21}$$

which is the expression usually reported in literature, because the EIP does not modify the time component of the current.

After integration on the whole plane of Eq. (4.1.21), and taking into account that the CS term is dominant over the Maxwell term at long distance [51, 115], we obtain the important property that every configuration with charge $Q = e \int \rho(t,\, \boldsymbol{x})\, d^2x$ transports also a magnetic flux $\Phi = \int B(t,\, \boldsymbol{x})\, d^2x$:

$$-g\,\Phi = Q \ . \tag{4.1.22}$$

This relation suggests the following interpretation: The system described by the Lagrangian (4.1.1) can be interpreted as analogous to a system of magnetic monopoles in (2+1) dimension obeying to a generalized exclusion-inclusion principle in the configuration space.
Finally, the spatial component of (4.1.16) is the Ampère law in $(2+1)$ dimensions with CS contribution:

$$\gamma\,\boldsymbol{\nabla}\wedge B - \frac{\gamma}{c}\,\frac{\partial\,\boldsymbol{E}}{\partial\,t} - g\,\boldsymbol{E}^* = \frac{e}{c}\,\boldsymbol{J} \ , \tag{4.1.23}$$

where $\boldsymbol{E}^*$ is the dual vector of the electric field with components $E_i^* = \epsilon_{ij}\,E_j$.

4.2 Hamiltonian formulation

The Hamiltonian formulation is crucial for the construction of self-dual solution which we have analyzed in the last chapter and for a variety of other purposes. In presence of gauge fields the Hamiltonian formulation reserves a special treatment. In fact, because of the non vanishing Maxwell term ($\gamma \neq 0$), the Lagrangian (4.1.3), (4.1.5) is degenerate in the velocity and the system can be described only as a constrained Hamiltonian, the constraint being given by the Gauss law. In its treatment we follow the Dirac-Bergmann approach [81].

The field $\pi_{_\phi}$ canonically conjugated of the field ϕ, is given by Eq. (1.3.17): $\pi_{_\phi} = \partial\mathcal{L}/\partial\dot{\phi}$ where the dot indicates the time derivative. Taking into account the expression of the Lagrangian given by Eqs. (4.1.3), (4.1.5) and setting for ϕ the fields ψ, ψ^*, A_μ, we obtain the following expressions:

$$\pi_{_\psi} = i\,\frac{\hbar}{2}\,\psi^* \ , \tag{4.2.1}$$

$$\pi_{_{\psi^*}} = -i\,\frac{\hbar}{2}\,\psi \ , \tag{4.2.2}$$

$$\Pi_\mu = \frac{\gamma}{c}F_{\mu 0} + \frac{g}{2\,c}\epsilon_{0\mu\nu}\,A^\nu \ . \tag{4.2.3}$$

Eqs. (4.2.1) and (4.2.2) give rise to primary derivative:

$$\xi_1 = \pi_{_\psi} - i\,\frac{\hbar}{2}\,\psi^* \ , \tag{4.2.4}$$

$$\xi_2 = \pi_{_{\psi^*}} + i\,\frac{\hbar}{2}\,\psi \ , \tag{4.2.5}$$

while Eq. (4.2.3) show that Π_0 vanishes and therefore it is a third primary constraint. Performing the Legendre transformation:

$$\mathcal{H} = \pi_{_\psi}\,\dot{\psi} + \pi_{_{\psi^*}}\,\dot{\psi}^* + \Pi_i\,\dot{A}^i - \mathcal{L} \ , \tag{4.2.6}$$

we obtain the Hamiltonian density of the system:

$$\mathcal{H} = \mathcal{H}_{\rm mat} + \mathcal{H}_{\rm gauge} \ , \tag{4.2.7}$$

where the Hamiltonian of the matter field $\mathcal{H}_{\rm mat}$ is:

$$\mathcal{H}_{\rm mat} = \frac{\hbar^2}{2\,m}\,|\boldsymbol{D}\psi|^2 - \kappa\,\frac{\hbar^2}{8\,m}\,[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*]^2 + \widetilde{U}(\psi^*\,\psi) + V\,\psi^*\,\psi \ , \tag{4.2.8}$$

while the one of the gauge field $\mathcal{H}_{\rm gauge}$ is:

$$\mathcal{H}_{\rm gauge} = \frac{\gamma}{2}\,(\boldsymbol{E}^2 + B^2) + A_0\,(g\,B - \gamma\,\boldsymbol{\nabla}\cdot\boldsymbol{E} + e\,\rho) + \partial_i\,(A_0\,\Pi^i) \ . \tag{4.2.9}$$

Let us introduce now the Poisson brackets (cfr. section 1.3) between two functionals:

$$\begin{aligned}
\{f(\boldsymbol{x}),\,g(\boldsymbol{y})\} \;=\; & \int\left[\frac{\delta\,f(\boldsymbol{x})}{\delta\,\psi(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\pi_{_\psi}(\boldsymbol{z})} - \frac{\delta\,f(\boldsymbol{x})}{\delta\,\pi_{_\psi}(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\psi(\boldsymbol{z})}\right] d^2z \\
+ & \int\left[\frac{\delta\,f(\boldsymbol{x})}{\delta\,\psi^*(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\pi_{_{\psi^*}}(\boldsymbol{z})} - \frac{\delta\,f(\boldsymbol{x})}{\delta\,\pi_{_{\psi^*}}(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\psi^*(\boldsymbol{z})}\right] d^2z \\
+ & \int\left[\frac{\delta\,f(\boldsymbol{x})}{\delta\,A^\mu(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,\Pi_\mu(\boldsymbol{z})} - \frac{\delta\,f(\boldsymbol{x})}{\delta\,\Pi_\mu(\boldsymbol{z})}\frac{\delta\,g(\boldsymbol{y})}{\delta\,A^\mu(\boldsymbol{z})}\right] d^2z \ .
\end{aligned} \tag{4.2.10}$$

The second class primary derivative (4.2.4) and (4.2.5) satisfy the relation:

$$\{\xi_1(t,\,\boldsymbol{x}),\,\xi_2(t,\,\boldsymbol{y})\} = -i\,\hbar\,\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}) \tag{4.2.11}$$

and can be accommodated by the introduction of the Dirac brackets:

$$\begin{aligned}
\{f(\boldsymbol{x}),\,g(\boldsymbol{y})\}_D = \{f(\boldsymbol{x}),\,g(\boldsymbol{y})\} \;& +\; \frac{i}{\hbar}\int\{f(\boldsymbol{x}),\,\xi_1(\boldsymbol{z})\}\,\{\xi_2(\boldsymbol{z}),\,g(\boldsymbol{y})\}\,d^2z \\
& -\; \frac{i}{\hbar}\int\{f(\boldsymbol{x}),\,\xi_2(\boldsymbol{z})\}\,\{\xi_1(\boldsymbol{z}),\,g(\boldsymbol{y})\}\,d^2z \ .
\end{aligned} \tag{4.2.12}$$

Expression (4.2.12) can be simplified if one solves the relation (4.2.4), (4.2.5) for π_{ψ} and π_{ψ^*} and treats f and g as functionals of ψ and ψ^* only. A straightforward calculations give:

$$\begin{aligned}\{f(\boldsymbol{x}),\, g(\boldsymbol{y})\}_D = & -\frac{i}{\hbar}\int\left[\frac{\delta\, f(\boldsymbol{x})}{\delta\,\psi(\boldsymbol{z})}\frac{\delta\, g(\boldsymbol{y})}{\delta\,\psi^*(\boldsymbol{z})} - \frac{\delta\, f(\boldsymbol{x})}{\delta\,\psi^*(\boldsymbol{z})}\frac{\delta\, g(\boldsymbol{y})}{\delta\,\psi(\boldsymbol{z})}\right] d^2z \\ & + \int\left[\frac{\delta\, f(\boldsymbol{x})}{\delta\, A^\mu(\boldsymbol{z})}\frac{\delta\, g(\boldsymbol{y})}{\delta\,\Pi_\mu(\boldsymbol{z})} - \frac{\delta\, f(\boldsymbol{x})}{\delta\,\Pi_\mu(\boldsymbol{z})}\frac{\delta\, g(\boldsymbol{y})}{\delta\, A^\mu(\boldsymbol{z})}\right] d^2z \,. \quad (4.2.13)\end{aligned}$$

The constraint $\Pi_0 = 0$ is first class and the requirement of its conservation leads to a secondary constraint:

$$\eta = \{\Pi_0,\, H\} = \gamma\,\boldsymbol{\nabla}\cdot\boldsymbol{E} - g\,B - e\,\rho = 0 \;, \quad (4.2.14)$$

where $H = \int \mathcal{H}\, d^2x$ is the Hamiltonian of the system given by (4.2.7). The secondary constraint $\eta = 0$ does not involve any further constraint since, how is easy to verify, $\{\eta,\, H\}_D = 0$.
The total Hamiltonian is now $H_T = H + \int \lambda\,\Pi_0\, d^2x$, where λ is a Lagrange multiplier. How it is well known, A_0 and Π_0 have no physical meaning, and $\Pi_0 = 0$ for all the time, while A_0 can take arbitrary values. Accordingly we may drop them out from the set of dynamical variables of the models. This can be accomplished by discarding the term $\lambda\,\Pi_0$ in H_T, the only role of which is to let A_0 vary arbitrarily, and by treating A_0 as an arbitrary multiplier. As a result, the Hamiltonian of the gauge fields $\mathcal{H}_{\rm gauge}$ becomes:

$$\mathcal{H}_{\rm gauge} = \frac{\gamma}{2}\,(\boldsymbol{E}^2 + B^2) \;, \quad (4.2.15)$$

and the total Hamiltonian density $\mathcal{H}_{\rm mat} + \mathcal{H}_{\rm gauge}$ is given in terms of canonical fields ψ, ψ^*, A_i, Π_i with $i = 1,\, 2$. We note that in $\mathcal{H}_{\rm gauge}$ the coupling constant g, introduced by the CS interaction, does not appear.
So far we have assumed $\gamma \neq 0$. In the pure CS model ($\gamma = 0$) the situation is somewhat different. The theory has two more second class primary constraint:

$$\xi_3 = \Pi_1 + \frac{g}{2\,c}\,A_2 = 0 \;, \quad (4.2.16)$$

$$\xi_4 = \Pi_2 - \frac{g}{2\,c}\,A_1 = 0 \;, \quad (4.2.17)$$

resulting from the definition of momenta Π_i. As

$$\{\xi_3(t,\,\boldsymbol{x}),\, \xi_4(t,\,\boldsymbol{y})\} = \frac{g}{c}\,\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}) \;, \quad (4.2.18)$$

these can be accommodated by modifying the brackets:

$$
\begin{aligned}
&\{f(\boldsymbol{x}),\, g(\boldsymbol{y})\}_D = \{f(\boldsymbol{x}),\, g(\boldsymbol{y})\} \\
&+\frac{i}{\hbar}\int \left[\{f(\boldsymbol{x}),\, \xi_1(\boldsymbol{z})\}\,\{\xi_2(\boldsymbol{z}),\, g(\boldsymbol{y})\} - \{f(\boldsymbol{x}),\, \xi_2(\boldsymbol{z})\}\,\{\xi_1(\boldsymbol{z}),\, g(\boldsymbol{y})\}\right] d^2z \\
&-\frac{c}{g}\int \left[\{f(\boldsymbol{x}),\, \xi_3(\boldsymbol{z})\}\,\{\xi_4(\boldsymbol{z}),\, g(\boldsymbol{y})\} - \{f(\boldsymbol{x}),\, \xi_4(\boldsymbol{z})\}\,\{\xi_3(\boldsymbol{z}),\, g(\boldsymbol{y})\}\right] d^2z \,. \quad (4.2.19)
\end{aligned}
$$

By solving Eqs. (4.2.16) and (4.2.17) for Π_1 and Π_2, this can be trough to the form:

$$
\begin{aligned}
\{f(\boldsymbol{x}),\, g(\boldsymbol{y})\} = &- \frac{i}{\hbar}\int \left[\frac{\delta f(\boldsymbol{x})}{\delta \psi(\boldsymbol{z})}\frac{\delta g(\boldsymbol{y})}{\delta \psi^*(\boldsymbol{z})} - \frac{\delta f(\boldsymbol{x})}{\delta \psi^*(\boldsymbol{z})}\frac{\delta g(\boldsymbol{y})}{\delta \psi(\boldsymbol{z})}\right] d^2z \\
&+ \int \left[\frac{\delta f(\boldsymbol{x})}{\delta A_1(\boldsymbol{z})}\frac{\delta g(\boldsymbol{y})}{\delta A_2(\boldsymbol{z})} - \frac{\delta f(\boldsymbol{x})}{\delta A_2(\boldsymbol{z})}\frac{\delta g(\boldsymbol{y})}{\delta A_1(\boldsymbol{z})}\right] d^2z \,, \quad (4.2.20)
\end{aligned}
$$

where f and g are considered as functionals of only ψ, ψ^*, A_1 and A_2.
In the pure CS case a further reduction is possible. In fact as we will show in the next section, one can explicitly solve the derivative (4.2.16) and (4.2.17) and end with ψ and ψ^* as the only canonical variables. In the general case, however, the fact that the constraint involves both Π_i and A_i makes such reduction impossible. Using the Poisson brackets "*naked*" from the constraint we can evaluate the evolution equations of the fields ψ and ψ^*:

$$
i\hbar\,\frac{\partial \psi}{\partial t} = \frac{\delta H}{\delta \psi^*} \,, \qquad (4.2.21)
$$

$$
i\hbar\,\frac{\partial \psi^*}{\partial t} = -\frac{\delta H}{\delta \psi} \,, \qquad (4.2.22)
$$

that are equal to Eq. (4.1.14) and its conjugate. The analogous equations for the fields A_i and Π_i become:

$$
\frac{\partial A^i}{\partial t} = \frac{\delta H}{\delta \Pi_i} \,, \qquad (4.2.23)
$$

$$
\frac{\partial \Pi_i}{\partial t} = -\frac{\delta H}{\delta A^i} \,, \qquad (4.2.24)
$$

and correspond respectively to Eq. (4.2.3) and Eq. (4.1.16).

Let us now define the following quantities:

$$
N = \int \psi^*\,\psi\, d^2x \,, \qquad (4.2.25)
$$

$$
<\boldsymbol{x}_c> = \frac{1}{N}\int \psi^*\,\boldsymbol{x}\,\psi\, d^2x \,, \qquad (4.2.26)
$$

$$< \boldsymbol{P} >= -i\,\frac{\hbar}{2}\int\left[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*\right]\,d^2x\ , \tag{4.2.27}$$

$$< M >= -i\,\frac{\hbar}{2}\int\boldsymbol{x}\wedge\left[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*\right]\,d^2x\ , \tag{4.2.28}$$

$$E = \int\mathcal{H}\,d^2x\ , \tag{4.2.29}$$

that represent, respectively, the particle number, the position of mass center, the linear momentum, the angular momentum and the energy of the system. The time evolution of these quantities is given by:

$$\frac{d\,f}{d\,t} = \{f,\,H\}_D + \frac{\partial\,f}{\partial\,t}\ , \tag{4.2.30}$$

for a generic functional $f(\psi,\,\psi^*,\,A_i,\,\Pi_i)$, where the last term in the right hand side takes into account the explicit time dependence of the functional f. After the computation we are left with the following Ehrenfest relations:

$$\frac{d}{d\,t}\,N = 0\ , \tag{4.2.31}$$

$$\frac{d}{d\,t} < \boldsymbol{x}_c >= \frac{1}{N}\int\boldsymbol{J}\,d^2x\ , \tag{4.2.32}$$

$$\frac{d}{d\,t} < \boldsymbol{P} >= \int\boldsymbol{f}\,d^2x - \int\psi^*\,\boldsymbol{\nabla}V\,\psi\,d^2x\ , \tag{4.2.33}$$

$$\frac{d}{d\,t} < M >= \int\boldsymbol{x}\wedge\boldsymbol{f}\,d^2x - \int\psi^*\,\left(\boldsymbol{x}\wedge\boldsymbol{\nabla}V\right)\,\psi\,d^2x\ ,$$

$$\frac{d}{d\,t}\,E = \int\psi^*\,\frac{d\,V}{d\,t}\,\psi\,d^2x\ , \tag{4.2.34}$$

where the quantity:

$$\boldsymbol{f} = e\,\rho + \frac{e}{c}\,\boldsymbol{J}\wedge\,B \tag{4.2.35}$$

is the density of the Lorentz force with EIP. We note that Eqs. (4.2.32)-(4.2.34) are formally identical to the analogous relations of the theory without the EIP ($\kappa = 0$). The difference is in the expression of the current: $\boldsymbol{J} = \boldsymbol{J}_0\,(1 + \kappa\,\rho)$. From Eq. (4.2.31) we see that the number of particles of the system is a constant of motion and from (4.2.34) we see that for external time-independent potential also the energy is preserved. Note also that EIP potential does not appear explicitly in the relations (4.2.33) and (4.2.34) as shown in section 2.3. The same statement is true for the nonlinear potential $\widetilde{U}(\rho)$ when it has a good behavior at infinity. Without the external force the linear and angular momentum are constants of motion. In the presence of a central force, while the linear momentum is not conserved, we have the

conservation of the angular momentum. Finally, it is trivial to note that from Eqs. (4.1.22) and (4.2.31) we have the conservation of charge and magnetic flux attached to the system.

4.2.1 Derivation of Ehrenfest relations

The Ehrenfest relations (4.2.31)-(4.2.34) for the observable (4.2.25)-(4.2.29) can be obtained following the computation reported in section 2.3.2 for the analogue quantity without the gauge coupling. The only quantities that involve more complicate steps are Eqs. (4.2.33) and (4.2.34). Therefore we derive in a detailed way only these two relations.

In the following we make the hypothesis that the fields vanish steeply at infinity and that we can neglect surface terms.
The first relation, Eq. (4.2.31), is a trivial consequence of the continuity equation (4.1.13), while the (4.2.32) is easily obtained from Eq. (4.1.13) as:

$$\frac{d}{d\,t} < \boldsymbol{x}_c > = \frac{1}{N}\int \boldsymbol{x}\,\frac{d\,\rho}{d\,t}\,d^2x = -\frac{1}{N}\int \boldsymbol{x}\,\boldsymbol{\nabla}\cdot\boldsymbol{J}\,d^2x = \frac{1}{N}\int \boldsymbol{J}\,d^2x\ , \qquad (4.2.36)$$

where we have performed an integration by parts in the last step.
Relation (4.2.34) is immediately obtained posing $f = E$ being E the Hamiltonian (4.2.29) and taking into account the property of the Poisson brackets $\{f,\,f\} = 0$ valid for every functional f. Eq. (4.2.30) becomes:

$$\frac{d\,f}{d\,t} = \frac{\partial\,f}{\partial\,t}\ , \qquad (4.2.37)$$

that is the relation (4.2.34), if we take into account that the explicit time dependence in the Hamiltonian can occur only in the external potential $V(t,\,\boldsymbol{x})$.

Let us consider now Eq. (4.2.33) for the component i of the momentum. Using Eqs. (4.2.30), (4.2.13) for $f =< P^i >$ given by (4.2.27) and taking into account the following functional derivative:

$$\frac{\delta f}{\delta\,\psi} = -i\hbar\,(D^i\psi)^*\ , \qquad (4.2.38)$$

$$\begin{aligned}\frac{\delta H}{\delta\,\psi^*} &= \frac{\hbar^2}{2\,m}\,D_j\,D^j\psi + \kappa\,\frac{\hbar^2}{2\,m}\left[\psi^*\,D^j\psi - \psi\,(D^j\psi)^*\right]\,D_j\psi \\ &+ \kappa\,\frac{\hbar^2}{4\,m}\,D_j\left[\psi^*\,D^j\psi - \psi\,(D^j\psi)^*\right]\,\psi + F(\rho)\,\psi + V\,\psi\ , \end{aligned} \qquad (4.2.39)$$

$$\frac{\delta f}{\delta A^i} = -\frac{e}{c}\,\rho\ , \tag{4.2.40}$$

$$\frac{\delta f}{\delta \Pi_i} = 0\ , \tag{4.2.41}$$

$$\frac{\delta H}{\delta \Pi_i} = -c\,F^{i0}\ , \tag{4.2.42}$$

we obtain:

$$\begin{aligned}\frac{d}{d\,t} < P^i > &= \int\Bigg\{-\frac{\hbar^2}{2\,m}\Big[(D^i\psi)^*\,D_jD^j\psi + D^i\psi\,(D_jD^j\psi)^*\Big]\\ &+ \kappa\,\frac{\hbar^2}{2\,m}\Big[\psi^*\,D^j\psi - \psi\,(D^j\psi)^*\Big]\Big[(D_j\psi)^*\,D^i\psi - D_j\psi\,(D^i\psi)^*\Big]\\ &+ \kappa\,\frac{\hbar^2}{4\,m}\,D_j\,\Big[\psi^*\,D^j\psi - \psi\,(D^j\psi)^*\Big]\Big[\psi^*\,D^i\psi - \psi\,(D^i\psi)^*\Big]\\ &- [F(\rho)+V]\;\partial^i\rho + e\,\rho\,F^{i0}\Bigg\}\,d^2x\ .\end{aligned} \tag{4.2.43}$$

Integrating by parts two times the first term and one time the third in the right hand side of (4.2.43) we obtain:

$$\begin{aligned}&\frac{d}{d\,t} < P^i >= \int\Bigg\{-\frac{\hbar^2}{2\,m}\Big[(D_jD^jD^i\psi)^*\,\psi + D^i\psi\,(D_jD^j\psi)^*\Big]\\ &+\kappa\,\frac{\hbar^2}{4\,m}\Big[\psi^*\,D^j\psi - \psi\,(D^j\psi)^*\Big]\ \Big[(D_j\psi)^*\,D^i\psi - D_j\psi\,(D^i\psi)^* - \psi^*D_j{}^i\psi + \psi\,(D_j{}^i\psi)^*\Big]\\ &-[F(\rho)+V]\;\partial^i\rho + e\,\rho\,F^{i0}\Bigg\}\,d^2x\ .\end{aligned} \tag{4.2.44}$$

Starting from the identity:

$$\begin{aligned}(D_jD^jD^i\psi)^*\,\psi + D^i\psi\,(D_jD^j\psi)^* &= (D^iD_jD^j\psi)^*\,\psi + \Big([D_j,\,D^i]\,D^j\psi\Big)^*\,\psi\\ &+ \Big(D^j[D_j,\,D^i]\psi\Big)^*\,\psi + D^i\psi\,(D_jD^j\psi)^*\end{aligned} \tag{4.2.45}$$

integrating by parts the third term in the right hand side of (4.2.45) and noting that:

$$[D_j,\,D^i] = -\frac{i\,e}{\hbar\,c}\Big(\partial_jA^i - \partial^iA_j\Big)\ , \tag{4.2.46}$$

we have:

$$(D_jD^jD^i\psi)^*\,\psi + D^i\psi\,(D_jD^j\psi)^* = \frac{i\,e}{\hbar\,c}\Big(\partial_jA^i - \partial^iA_j\Big)\ \Big[\psi^*\,D^j\psi - \psi\,(D^j\psi)^*\Big]\ , \tag{4.2.47}$$

and moreover:

$$\left[(D_j\psi)^*\, D^i\psi - D_j\psi\,(D^i\psi)^* - \psi^* D_j{}^i\psi + \psi\,(D_j{}^i\psi)^*\right] = -i\,\frac{2\,e}{\hbar\,c}\left(\partial_j A^i - \partial^i A_j\right)\,\rho\ . \quad (4.2.48)$$

Inserting relations (4.2.47), (4.2.48) in (4.2.44) we can obtain the relation:

$$\begin{aligned}\frac{d}{d\,t} < P^i > \;&=\; \int\left\{-\frac{i\,e\,\hbar}{2\,m\,c}\left(\partial_j A^i - \partial^i A_j\right)\left[\psi^*\, D^j\psi - \psi\,(D^j\psi)^*\right]\,(1+\kappa\,\rho) + e\,\rho\,F^{i0}\right\} d^2x \\ &+\; \int \rho\,\partial^i\,F(\rho)\,d^2x + \int \rho\,\partial^i\,V\,d^2x\ . \end{aligned} \quad (4.2.49)$$

By taking into account the relation $F = d\widetilde{U}/d\rho$, the last two integrals in (4.2.49) can be written as:

$$\int \partial^i\,(\rho\,F - \widetilde{U})\,d^2x\ , \quad (4.2.50)$$

and therefore, if the potential and the field ρ have a good behavior at infinity, it can be ignored. Then Eq. (4.2.49) is equals Eq. (4.2.33) if we take into account the definition of electric and magnetic fields and of current (4.1.11).
Eq. (4.2.34) can be obtained following the same iter used to deduce the relation (4.2.33), we leave the easy but tedious computation to the reader.

4.3 Conservation laws

Let us set, in the Ehrenfest relations (4.2.31)-(4.2.34), the external potential $V = 0$, we can deduce the following conserved quantities:

$$\frac{d}{d\,t}\,N = 0\ , \qquad N = \int \rho\,d^2x\ , \quad (4.3.1)$$

$$\frac{d}{d\,t}\boldsymbol{P}_{\rm tot} = 0\ , \qquad \boldsymbol{P}_{\rm tot} =< \boldsymbol{P} > + \int \boldsymbol{E}\wedge B\,d^2x\ , \quad (4.3.2)$$

$$\frac{d}{d\,t}M_{\rm tot} = 0\ , \qquad M_{\rm tot} =< M > + \int \boldsymbol{x}\wedge(\boldsymbol{E}\wedge B)\,d^2x\ , \quad (4.3.3)$$

$$\frac{d}{d\,t}\,E = 0\ , \qquad E = \int \mathcal{H}\,d^2x\ , \quad (4.3.4)$$

N is the particle number, $\boldsymbol{P}_{\rm tot}$ is the total linear momentum, $M_{\rm tot}$ the total angular momentum and finally E is the total energy of the system.
In this section we want to obtain the above constants of motion (4.3.1)-(4.3.4) by means of an approach different from that used in the previous section, i.e. by means of fundamental principles and of the Nöther theorem. The deduction of these

quantities is close to the content of chapter III to obtain the energy-momentum tensor for the EIP-Schrödinger equation, but here, the presence of the CS term makes the iter a little more complicate.

We consider the symmetry related to the invariance of the system over space-time translations. If we consider an infinitesimal transformation:

$$t \to t - a\ , \qquad \boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{a}\ , \tag{4.3.5}$$

it is easy to see that the variations of the fields are given by:

$$\delta_a\psi = a^\mu\,\partial_\mu\psi, \qquad \delta_a\psi^* = a^\mu\,\partial_\mu\psi^*, \qquad \delta_a A_\nu = a^\mu\,\partial_\mu A_\nu\ , \tag{4.3.6}$$

and the function $\mathcal{L}$ changes for a quantity $\delta\,\mathcal{L} = \delta^{\mu\nu}\partial_\nu\mathcal{L}$, (here there is no sum over the repeated index ν); where the symbol $\delta^{\mu\nu}$ is the Kronecker tensor and the index μ selects the variation of $\mathcal{L}$ over space translations from time translations.
Introducing the tensor $T^{\mu\nu}$, defined as:

$$T^{i\nu} = \mathcal{J}^\nu_{\text{space}}\ , \tag{4.3.7}$$

where $\mathcal{J}^\nu_{\text{space}}$ is the Nöther current generated from a space translation, and

$$T^{0\nu} = \mathcal{J}^\nu_{\text{time}}\ , \tag{4.3.8}$$

where now $\mathcal{J}^\nu_{\text{time}}$ is the Nöther current generated from a time translation. Then, from the expression of the density of Lagrangian (4.1.1) and using the Nöther formula:

$$\mathcal{J}^\nu = \frac{\partial\,\mathcal{L}}{\partial\,(\partial_\nu\psi)}\,\delta\,\psi + \frac{\partial\,\mathcal{L}}{\partial\,(\partial_\nu\psi^*)}\,\delta\,\psi^* + \frac{\partial\,\mathcal{L}}{\partial\,(\partial_\nu A_\mu)}\,\delta\,A_\mu - \delta^{\mu\nu}\,\partial_\mu\mathcal{L}\ , \tag{4.3.9}$$

and the definitions (4.3.7) and (4.3.8) we obtain the quantities:

$$\begin{aligned}
\widetilde{T}^{00} &= \frac{\hbar^2}{2\,m}|\boldsymbol{D}\psi|^2 - \kappa\,\frac{\hbar^2}{8m}\,[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*]^2 + \widetilde{U}(\rho) + e\,A^0\,\rho \\
&+ \gamma\left(\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} - F^{0\tau}\,\partial^0 A_\tau\right) - \frac{g}{2}\left(\frac{1}{2}\epsilon^{\tau\mu\nu}A_\tau\,F_{\mu\nu} - \epsilon^{0\lambda\tau}\,A_\lambda\,\partial^0 A_\tau\right)\ ,
\end{aligned} \tag{4.3.10}$$

$$\widetilde{T}^{0i} = i\,c\,\frac{\hbar}{2}\left(\psi^*\,\partial^i\psi - \psi\,\partial^i\psi^*\right) - \gamma\,F^{0\tau}\,\partial^i A_\tau - \frac{g}{2}\,\epsilon^{0\lambda\tau}\,A_\lambda\,\partial^i A_\tau\ , \tag{4.3.11}$$

$$\begin{aligned}
\widetilde{T}^{i0} &= \frac{\hbar^2}{2\,m}\left[(D^i\psi)^*\,\partial^0\psi + D^i\psi\,\partial^0\psi^*\right] - \kappa\,\frac{\hbar^2}{4\,m}\,\left[\psi^*\,D^i\psi - \psi\,(D^i\psi)^*\right]\left(\psi^*\,\partial^0\psi - \psi\,\partial^0\psi^*\right) \\
&- \gamma\,F^{i\tau}\,\partial^0 A_\tau - \frac{g}{2}\,\epsilon^{i\lambda\tau}\,A_\lambda\,\partial^0 A_\tau\ ,
\end{aligned} \tag{4.3.12}$$

$$
\begin{aligned}
\widetilde{T}^{ij} &= \frac{\hbar^2}{2\,m}\left[(D^i\psi)^*\,\partial^j\psi + D^i\psi\,\partial^j\psi^*\right] - \kappa\,\frac{\hbar^2}{4\,m}\left[\psi^*\,D^i\psi - \psi\,(D^i\psi)^*\right]\left(\psi^*\,\partial^j\psi - \psi\,\partial^j\psi^*\right) \\
&- \gamma\,F^{i\tau}\,\partial^j\,A_\tau - \frac{g}{2}\,\epsilon^{i\lambda\tau}\,A_\lambda\,\partial^j\,A_\tau + \delta^{ij}\left\{i\,c\,\frac{\hbar}{2}\,[\psi^*\,D_0\psi - \psi\,(D_0\psi)^*] - \frac{\hbar^2}{2\,m}\,|\boldsymbol{D}\psi|^2\right. \\
&+ \left.\kappa\,\frac{\hbar^2}{4m}\,[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi)^*]^2 - \widetilde{U}(\rho) - \frac{\gamma}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{g}{4}\epsilon^{\tau\mu\nu}\,A_\tau\,F_{\mu\nu}\right\}\ . \qquad (4.3.13)
\end{aligned}
$$

In (4.3.10)-(4.3.13) the gauge invariance of the tensor $\widetilde{T}^{\mu\nu}$ does not appear explicitly. We consider now the new tensor $\widetilde{\widetilde{T}}^{\mu\nu}$ given by:

$$
\widetilde{\widetilde{T}}^{\mu\nu} = \widetilde{T}^{\mu\nu} + \widetilde{f}^{\mu\nu}\ , \qquad (4.3.14)
$$

where the quantity:

$$
\widetilde{f}^{\mu\nu} = A^\nu\left(\gamma\,\partial_\tau F^{\tau\mu} + \frac{g}{2}\,\epsilon^{\mu\lambda\tau}\,F_{\lambda\tau} - \frac{e}{c}\,J^\mu\right)\ , \qquad (4.3.15)
$$

vanishes because of the motion equations (4.1.16). We observe that the tensor $\widetilde{\widetilde{T}}^{\mu\nu}$ is gauge invariant. Finally, if we remember that the expression of the energy-momentum tensor is defined modulo a quantity $\partial_\tau\,X^{\tau\mu\nu}$ where $X^{\tau\mu\nu}$ is a third rank tensor which is antisimetric in the two indices τ and μ given by:

$$
X^{\tau\mu\nu} \equiv -\gamma\,F^{\tau\mu}\,A^\nu + \frac{g}{2}\,\epsilon^{\tau\mu\lambda}\,A_\lambda\,A^\nu\ , \qquad (4.3.16)
$$

it is easy to verify that the tensor $T^{\mu\nu}$ defined as:

$$
T^{\mu\nu} = \widetilde{\widetilde{T}}^{\mu\nu} - \gamma\,\partial_\tau\,(F^{\tau\mu}\,A^\nu) + \frac{g}{2}\,\epsilon^{\tau\mu\lambda}\,\partial_\tau\,(A^\nu\,A_\lambda)\ , \qquad (4.3.17)
$$

is gauge invariant and obeys to the continuity equation:

$$
\partial_\mu\,T^{\mu\nu} \equiv \partial_\mu\,\widetilde{\widetilde{T}}^{\mu\nu} = 0\ . \qquad (4.3.18)
$$

As a consequence of (4.3.17) the motion constants of the system:

$$
\int T^{\mu 0}\,d^2x = \int \widetilde{\widetilde{T}}^{\mu 0}\,d^2x\ , \qquad (4.3.19)
$$

are not modified by the substitutions (4.3.14) and (4.3.17). The final expression of the components of the energy-momentum tensor $T^{\mu\nu}$ are:

$$
T^{00} = \frac{\hbar^2}{2\,m}|\boldsymbol{D}\psi|^2 - \kappa\,\frac{\hbar^2}{8m}\,[\psi^*\,\boldsymbol{D}\,\psi - \psi\,(\boldsymbol{D}\,\psi)^*]^2 + \widetilde{U}(\rho)
$$

$$
\begin{aligned}
&+ \quad \frac{\gamma}{4} F_{ij}\, F^{ij} - \frac{\gamma}{2} F_{0i}\, F^{0i} \ , && (4.3.20) \\
T^{0i} &= \quad i\, c\, \frac{\hbar}{2} \left[\psi^*\, D^i \psi - \psi\, (D^{\ i} \psi)^* \right] - \gamma\, F^{0j}\, F^i_{\ j} \ , && (4.3.21) \\
T^{i0} &= \quad \frac{\hbar^2}{2\, m} \left[(D^i \psi)^*\, D^0 \psi + D^i \psi\, (D^0 \psi)^* \right] \\
&- \quad \kappa\, \frac{\hbar^2}{4\, m} \left[\psi^*\, D^i \psi - \psi\, (D^i \psi)^* \right] \left[\psi^*\, D^0 \psi - \psi\, (D^0 \psi)^* \right] - \gamma\, F^{0j}\, F^i_{\ j} && (4.3.22) \\
T^{ij} &= \quad \frac{\hbar^2}{2\, m} \left[(D^i \psi)^*\, D^j \psi + D^i \psi\, (D^j \psi)^* \right] \\
&- \quad \kappa\, \frac{\hbar^2}{4\, m} \left[\psi^*\, D^i \psi - \psi\, (D^i \psi)^* \right] \left[\psi^*\, D^j \psi - \psi\, (D^j \psi)^* \right] \\
&- \quad \delta^{ij} \left\{ \frac{\hbar^2}{4\, m} \Delta \rho + \kappa\, \frac{\hbar^2}{8m} \left[\psi^*\, \boldsymbol{D}\, \psi - \psi\, (\boldsymbol{D}\, \psi)^* \right]^2 \right. \\
&+ \quad \left. \widetilde{U}(\rho) - \rho\, \frac{d\, \widetilde{U}(\rho)}{d\, \rho} \right\} - \gamma\, F^{i\lambda}\, F^j_{\ \lambda} - \frac{\gamma}{4} \delta^{ij}\, F^{\mu\nu}\, F_{\mu\nu} \ . && (4.3.23)
\end{aligned}
$$

We note that in the above expression of $T^{\mu\nu}$ the CS contribution does not appear explicitly. This is due to the topological nature of the CS interaction. How we have noted in the introduction of this chapter, independently of the geometry of the variety in which the system is imbedded, the expression of the CS Lagrangian appears the same without the necessity to introduce the metric tensor $g_{\mu\nu}$. In fact the quantity $\epsilon_{\mu\nu\lambda}$ is a good tensor. As a consequence, if we take into account that the $T^{\mu\nu}$ tensor can be also obtained by a functional variation of the Lagrangian density with respect to the metric we obtain immediately the statement that the contributes of CS to the energy-momentum tensor are null due to the independence of its from $g_{\mu\nu}$. Notwithstanding the presence of the CS term modifies the form of the gauge fields A_μ because they must satisfies Eq. (4.1.16).

The tensor $T^{\mu\nu}$ is symmetric only in the spatial indices $T^{ij} = T^{ji}$, because the theory is not Lorentz invariant but only rotation invariant and satisfies the continuity equation:

$$
\partial_\mu\, T^{\mu\,\nu} = 0 \ . \tag{4.3.24}
$$

Of course, the two following quantities are conserved:

$$
E = \int T^{00}\, d^2x \ , \qquad P^i_{\rm tot} = \frac{1}{c} \int T^{0i}\, d^2x \ . \tag{4.3.25}
$$

In section 3.2 we have shown that the energy density T^{00} is a semidefinite positive quantity. This properties hold also in presence of the gauge field. In fact, in the

case $\kappa > 0$ by using Eqs. (4.1.11), (4.3.20) and the expression of the fields $\boldsymbol{E}$ and B as functions of the potential A_μ, the quantity T^{00} assume the form

$$T^{00} = \frac{\hbar^2}{2\,m}|\boldsymbol{D}\psi|^2 + \kappa\,\frac{m}{2}\left(\frac{\boldsymbol{J}}{1+\kappa\,\rho}\right)^2 + \widetilde{U}(\rho) + \frac{\gamma}{2}\,(\boldsymbol{E}^2 + B^2)\ . \tag{4.3.26}$$

Therefore, if $\widetilde{U}(\rho) = 0$, T^{00} is a semidefinite positive quantity. In the case $\kappa < 0$, after rewriting the quantity T^{00} as:

$$T^{00} = \frac{\hbar^2}{2\,m}|\boldsymbol{D}\psi|^2\,(1+\kappa\rho) - \kappa\,\frac{\hbar^2}{8\,m}\,(\boldsymbol{\nabla}\,\rho)^2 + \widetilde{U}(\rho) + \frac{\gamma}{2}\,(\boldsymbol{E}^2 + B^2)\ , \tag{4.3.27}$$

we can obtain the same conclusion, if we take into account that $1+\kappa\,\rho \geq 0$.
We consider now the invariance of the system under spatial rotations defined by the transformations:

$$x^i \to \Omega^i{}_j\,x^j\ , \qquad \Omega^i{}_j\,\Omega^j{}_k = \delta^i{}_k\ . \tag{4.3.28}$$

The angular momentum density M^0 and the relative flux M^i form the vector $M^\mu = (M^0,\,\boldsymbol{M})$, that is obtained from the tensor $T^{\mu\nu}$ by the relation:

$$M^\mu = \epsilon_{ij}\,x^i\,T^{\mu j}\ . \tag{4.3.29}$$

From the symmetry $T^{ij} = T^{ji}$ we infer the continuity equation for M^μ:

$$\partial_\mu M^\mu = 0\ . \tag{4.3.30}$$

The total angular momentum:

$$M_{\rm tot} = \frac{1}{c}\int M^0\,d^2x\ , \tag{4.3.31}$$

is time conserved and, like $\boldsymbol{P}_{\rm tot}$, is not modified by the presence of EIP potential. The quantity (4.3.25) and (4.3.31) are the generators of the space-time translation and rotations supplemented by local gauge transformations when performed on gauge not invariant fields. In fact if we consider for example the quantity:

$$g = e^{\frac{i}{\hbar}\,a_i\,P^i}\ , \tag{4.3.32}$$

which generate a space translation with parameter a_i, the linear part of the variations of the fields can be performed trough the "*naked*" Poisson brackets (4.2.13):

$$\begin{aligned}\psi(t,\,\boldsymbol{x}) \to \psi_a(t,\,\boldsymbol{x}) &= g\,\psi(t,\,\boldsymbol{x}) = \psi(t,\,\boldsymbol{x}) + a_i\,\{\psi(t,\,\boldsymbol{x}),\,P^i\}_D \\ &= \psi(t,\,\boldsymbol{x}) + a_i\,\partial^i\psi(t,\,\boldsymbol{x}) + i\,e\,a_i\,\omega(t,\,\boldsymbol{x})\,\psi(t,\,\boldsymbol{x})\ ,\end{aligned} \tag{4.3.33}$$

$$\begin{aligned}\psi^*(t,\,\boldsymbol{x}) \to \psi^*_a(t,\,\boldsymbol{x}) &= g\,\psi^*(t,\,\boldsymbol{x}) = \psi^*(t,\,\boldsymbol{x}) + a_i\,\{\psi^*(t,\,\boldsymbol{x}),\,P^i\}_D \\ &= \psi^*(t,\,\boldsymbol{x}) + a_i\,\partial^i\psi^*(t,\,\boldsymbol{x}) + i\,e\,a_i\,\omega(t,\,\boldsymbol{x})\,\psi^*(t,\,\boldsymbol{x}) \quad (4.3.34)\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}(t,\,\boldsymbol{x}) \to \boldsymbol{A}(t,\,\boldsymbol{x}) &= g\,\boldsymbol{A}(t,\,\boldsymbol{x}) = \boldsymbol{A}(t,\,\boldsymbol{x}) + a_i\,\{\boldsymbol{A}(t,\,\boldsymbol{x}),\,P^i\}_D \\ &= \boldsymbol{A}(t,\,\boldsymbol{x}) + a_i\,\partial^i\boldsymbol{A}(t,\,\boldsymbol{x}) + i\,e\,a_i\,\omega(t,\,\boldsymbol{x})\,\boldsymbol{A}(t,\,\boldsymbol{x})\ , \quad (4.3.35)\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Pi}(t,\,\boldsymbol{x}) \to \boldsymbol{\Pi}(t,\,\boldsymbol{x}) &= g\,\boldsymbol{\Pi}(t,\,\boldsymbol{x}) = \boldsymbol{\Pi}(t,\,\boldsymbol{x}) + a_i\,\{\boldsymbol{\Pi}(t,\,\boldsymbol{x}),\,P^i\}_D \\ &= \boldsymbol{\Pi}(t,\,\boldsymbol{x}) + a_i\,\partial^i\boldsymbol{\Pi}(t,\,\boldsymbol{x}) + i\,e\,a_i\,\omega(t,\,\boldsymbol{x})\,\boldsymbol{\Pi}(t,\,\boldsymbol{x}) \quad (4.3.36)\end{aligned}$$

with ω the gauge parameter. The fact that the quantities (4.3.25) and (4.3.31) generate translations supplied by gauge transformations is due to structure of the Poisson brackets (4.2.13) which are obtained after taking into account the constraint (4.2.14) which is the generator of the gauge transformation (see below). However, if we restrict ourselves to gauge-invariant fields, e.g. $\rho(t,\,\boldsymbol{x})$, $\boldsymbol{E}(t,\,\boldsymbol{x})$, and $B(t,\,\boldsymbol{x})$, Eqs. (4.3.25), (4.3.31) generates pure translations:

$$\{\boldsymbol{P}(t,\,\boldsymbol{x}),\,\rho(t,\,\boldsymbol{x})\} = \boldsymbol{\nabla}\,\rho(t,\,\boldsymbol{x})\ , \quad (4.3.37)$$

$$\{\boldsymbol{P}(t,\,\boldsymbol{x}),\,F_{\mu\nu}(t,\,\boldsymbol{x})\} = \boldsymbol{\nabla}\,F_{\mu\nu}(t,\,\boldsymbol{x})\ . \quad (4.3.38)$$

It is known that the introduction of the CS interaction confers to the matter field a non conventional statistical behavior [51, 112]. This statistical behavior is the result of the Aharonov-Bohm effect [116], because, in presence of the CS term, each charge field carries forward, intrinsically, also point magnetic vortices. This statement is true also in the presence of the Maxwell term, because only the asymptotic behavior is invoked, which, because of the high-derivative terms of the Maxwell coupling, is dominated by the CS term. This anomalous situation in not modified by the presence of EIP because its does not modify the expression of $<\boldsymbol{P}>$ and $<M>$. In fact, posing for simplicity $\gamma = 0$, the second relation of (4.3.25) and (4.3.31) becomes:

$$<\boldsymbol{P}>= -\frac{i\,\hbar}{2}\int(\psi^*\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)\,d^2x - \frac{e}{c}\int\boldsymbol{A}\,\rho\,d^2x\ , \quad (4.3.39)$$

$$<M>= -\frac{i\,\hbar}{2}\int\boldsymbol{x}\wedge(\psi^*\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)\,d^2x - \frac{e}{c}\int\boldsymbol{x}\wedge\boldsymbol{A}\,\rho\,d^2x\ . \quad (4.3.40)$$

Eq. (4.1.21) can be solved without ambiguity in the Coulomb gauge ($\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$):

$$\boldsymbol{A}(\boldsymbol{x}) = -\frac{e}{2\,\pi\,g}\int\boldsymbol{\nabla}\cdot\arctan\left(\frac{x_2 - y_2}{x_1 - y_1}\right)\,\rho(\boldsymbol{y})\,d^2y\ , \quad (4.3.41)$$

and inserting it into Eqs. (4.3.39) and (4.3.40), after integration, we obtain:

$$< \boldsymbol{P} >= -\frac{i\,\hbar}{2}\int (\psi^*\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)\,d^2x \ , \tag{4.3.42}$$

$$< M >= -\frac{i\,\hbar}{2}\int \boldsymbol{x}\wedge(\psi^*\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)\,d^2x - \frac{Q^2}{4\,\pi\,c\,g} \ . \tag{4.3.43}$$

From Eq. (4.3.43) we can see that the angular momentum of the field ψ is the sum of two terms: the former represents the orbital angular momentum, while the last, the spin, is responsible of the anomalous behavior of the system. This result, already known, is not too surprising: its origin is in the Gauss law (4.1.21) that is not modified by the presence of EIP. As a consequence of Eq. (4.3.43) it follows that EIP does not change the anyonic behavior of the system and so that the spin-statistics relation of the anyonic systems holds. Therefore the Lagrangian density (4.1.3), (4.1.5) describe a model of anyonic particle with kinetic obeying to the EIP. This one generate an exclusion-inclusion effect in the configuration space wilts the CS term is responsible of the not conventional quantum statistics.

Finally, we discuss briefly the gauge transformations. In section 4.1 we have seen that the Lagrangian (4.1.3), (4.1.5) is invariant up to a divergence over global gauge $U(1)$ transformations:

$$\psi(t,\,\boldsymbol{x}) \ \rightarrow \ e^{-i(e/\hbar\,c)\,\omega}\,\psi(t,\,\boldsymbol{x}) \ , \tag{4.3.44}$$

$$A_\mu(t,\,\boldsymbol{x}) \ \rightarrow \ A_\mu(t,\,\boldsymbol{x}) \ , \tag{4.3.45}$$

with $\delta\mathcal{L} = 0$. The Nöther current is now:

$$\mathcal{J}^\mu = \left\{c\,\rho,\ -\frac{i\,\hbar}{2\,m}\,(1+\kappa\,\rho)\,[\psi^*\,\boldsymbol{D}\psi - \psi\,(\boldsymbol{D}\psi^*)]\right\} \ , \tag{4.3.46}$$

which satisfies the continuity equation (4.1.20).
We consider now the local gauge $U(1)$ transformation:

$$\psi(t,\,\boldsymbol{x}) \ \rightarrow \ e^{-i\,(e/\hbar\,c)\,\omega(t,\,\boldsymbol{x})}\,\psi(t,\,\boldsymbol{x}) \ , \tag{4.3.47}$$

$$A_\mu(t,\,\boldsymbol{x}) \ \rightarrow \ A_\mu(t,\boldsymbol{x}) + \partial_\mu\omega(t,\,\boldsymbol{x}) \ . \tag{4.3.48}$$

It is well known, that the Nöther theorem, in the case of symmetries related to continuous transformations, does not give conserved quantities but from it we obtain the same identities. In fact, the gauge charge associated with this continuous symmetry vanishes identically.
In the case of the transformations (4.1.17) and (4.1.18) we have (using the notations

of section 3.2):

$$\delta\,\psi = -\frac{i\,e}{\hbar\,c}\,\psi\,\omega\ , \tag{4.3.49}$$

$$\delta\,A_\mu = \partial_\mu\omega\ , \tag{4.3.50}$$

$$f^\mu = \frac{g}{4}\,\epsilon^{\mu\nu\tau}\,F_{\nu\tau}\,\omega\ . \tag{4.3.51}$$

The Nöther vector $\mathcal{J}^\mu \equiv (\mathcal{J}^0,\,\mathcal{J}^i)$ can be calculated by using (3.2.3), we obtain:

$$\mathcal{J}^0 = (e\,\rho - \gamma\,\boldsymbol{\nabla}\cdot\boldsymbol{E} + g\,B)\,\omega\ , \tag{4.3.52}$$

$$\mathcal{J}^i = \left(\frac{e}{c}\,J^i - \gamma\,\partial_\mu F^{\mu i} - \frac{g}{2}\,\epsilon^{i\mu\nu}F_{\mu\nu}\right)\,\omega\ . \tag{4.3.53}$$

By means of Eq. (3.2.4) the charge Q_ω, given by $Q_\omega = \int \mathcal{J}_0\,d^2x$, takes the expression:

$$Q_\omega = \int (\gamma\,\boldsymbol{\nabla}\cdot\boldsymbol{E} - g\,B - e\,\rho)\ \omega\,d^2x\ . \tag{4.3.54}$$

We can easily see, after using the Gauss theorem (4.1.21), that Q_ω is identically zero . This means that the Gauss condition (4.1.21) is satisfied at each time, during the motion.

Chapter V

Canonical Systems Obeying to EIP: Solitons

In the previous chapters we have considered the definition of collectively interacting particles obeying to EIP and we have studied the main properties of a particles system.
In this chapter and in the following one, we study a special class of states of these systems called *solitons*. Solitons are special solutions of nonlinear evolution equations, not obtainable within perturbative methods, that preserve their shape during the propagation. Soliton solutions appear in many topics in physics. For instance, the cubic Schrödinger equation solutions describe propagation of deep water waves and of modulated ion-acoustic waves in plasma, three-dimensional diffractive patterns of a laser beam [60] and more importantly describe the recently observed Bose-Einstein condensation in rarefied vapors of metal ^{7}Li, ^{23}Na ^{87}Rb [42, 117, 118, 119].
Due to their not spreading property and to their particle-like behavior, solitons can be used in field theory in order to describe elementary particles. Rigorously speaking, solitons are wave packets which preserve their shape during their propagation and after collision with other solitons. Usually it is required that solitons preserve asymptotically their speed after collisions, but this is hard to met with the interpretation of solitons as particles.
Typically, in literature, soliton term is misused to denote solitary waves. These are also wave packets preserving their shapes but no assumptions are made about their behavior after collisions. Thus, a single soliton is a solitary wave but solitary waves are not necessarily solitons. In order to see if a solution is a "*genuine*" soliton it is necessary to follow it after collision with another soliton requiring the knowledge of multi-solitons states. This solutions generally are very hard to be obtained.
In the following, we use the word soliton but all our solutions are simply solitary

waves.

5.1 Solitons

We study a particular class of solutions of Eq. (2.1.25) in the free case, i.e. when $V = 0$. In this situation, we can consider the motion of the mass center on a straight line with uniform velocity (for a discussion of solitons in an external potential see for example Ref. [120]). In addition to the condition $V = 0$, we limit our attention to the one dimensional case when the EIP holds. The extension to highest dimensions is not straightforward. In particular the easy case of waves with amplitude modulated only in one space-dimension is non physical situation because they carry infinite energy.
In chapter III we have studied the symmetry Lie group of the system obeying to the EIP. The result was that Eq. (2.1.25) is invariant over roto-translation transformations, scaling transformations and the global unitary $U(1)$ group transformations. We have a nine parameter full symmetry group. We can use this information to search special solutions obeying to EIP. In fact, we can require that the solution should be invariant under a proper selected subgroup of the full symmetry group so that the solution itself can be obtained solving an ordinary differential equation rather than a more complicate partial differential equation.
In this section we focus our attention on solutions that are constants in between the y and z spatial direction and moreover are invariant over the x and t translation:

$$x \to x \pm u\,\varepsilon\ , \qquad\qquad t \to t + \varepsilon\ , \tag{5.1.1}$$

with ε and u constants. The global invariant of this transformation is the quantity $x \mp u\,t$. Therefore we require that the field ψ depends only on the time t and on the coordinate of the soliton mass center $\xi = x \mp u\,t$ where u now has the meaning of velocity of the soliton (as usual, the sign minus stands for a soliton moving from the left to the right side of the x axis, while the sign plus stands for an antisoliton moving in the opposite versus). Thus the wave function becomes $\psi(x,\,t) \equiv \psi(\xi,\,t)$ where we include a possible explicit time dependence in the phase. (Rigorously speaking, this solution is not invariant over the transformation (5.1.1) because the time dependence on S. Notwithstanding, this assumption does not modify the general procedure to find soliton solutions, allowing to obtain a more general expression). In the following we use the method described in ref. [121], valid for the NLSEs that are most frequently encountered in physical problems. We assume that, for $\xi \to \pm\infty$, the particle density $\rho(\xi) \to 0$ so that $\int_{-\infty}^{+\infty} \rho(\xi)\,d\xi = N$, where N represents the collective

particle number contained in the soliton. Moreover, the phase S is written as:

$$S = s(\xi) - \epsilon\, t \ , \tag{5.1.2}$$

and the field ψ assumes the form:

$$\psi = \rho(\xi)^{1/2} \exp\left\{\frac{i}{\hbar}[s(\xi) - \epsilon\, t]\right\} \ . \tag{5.1.3}$$

It is now easy to verify that the *Hamilton-Jacobi* equations (2.2.1) and the continuity equation (2.2.2) describing the solitonic state can be reduced to the following system of coupling equations:

$$\pm u\,\frac{\partial s}{\partial \xi} = \frac{1 + 2\,\kappa\,\rho}{2\,m}\left(\frac{\partial s}{\partial \xi}\right)^2 + U_q(\rho) + F(\rho) - \epsilon \ , \tag{5.1.4}$$

$$\pm u\,\frac{\partial \rho}{\partial \xi} = \frac{1}{m}\frac{\partial}{\partial \xi}\left[\frac{\partial s}{\partial \xi}\,\rho\,(1 + \kappa\,\rho)\right] \ , \tag{5.1.5}$$

where we have indicated with $U_q(\rho)$ the one dimensional quantum potential in the ξ coordinate:

$$U_q(\rho) = -\frac{\hbar^2}{2\,m}\,\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial \xi^2} \ . \tag{5.1.6}$$

The quantum velocity $v_q(\xi) = m^{-1}\partial s(\xi)/\partial \xi$ must be finite when $\xi \to \pm\infty$; with this condition Eq. (5.1.5) can be integrated a first time, obtaining:

$$\frac{\partial\, s}{\partial\, \xi} = \pm\frac{m\,u}{1 + \kappa\,\rho} \ , \tag{5.1.7}$$

and after second integration with the condition $\xi(0) = 0$, we have:

$$s(\xi) = \pm m\,u \int\limits_0^\xi \frac{d\,\xi'}{1 + \kappa\,\rho(\xi')} \ . \tag{5.1.8}$$

Eq. (5.1.2) and (5.1.8) allow us to calculate the phase $S(\xi)$ provided that the quantity $\rho(\xi)$ is known.

To evaluate the density $\rho(\xi)$ we note that, if we take into account Eq. (5.1.7), Eq. (5.1.4) reduces to the following second order differential equation:

$$\frac{2}{\rho}\frac{d^2\,\rho}{d\,\xi^2} - \left(\frac{1}{\rho}\frac{d\,\rho}{d\,\xi}\right)^2 + \frac{(2\,m\,u/\hbar)^2}{(1 + \kappa\,\rho)^2} - \frac{8\,m}{\hbar^2}\,F(\rho) + \frac{8\,m\,\epsilon}{\hbar^2} = 0 \ . \tag{5.1.9}$$

Before solving this equation, we observe that ϵ can be written in the form:

$$\epsilon = U_q + F(\rho) - \frac{1}{2}\, m\, u^2 \frac{1}{(1+\kappa\,\rho)^2} \ . \tag{5.1.10}$$

Equation (5.1.10) has an immediate physical interpretation: the quantum potential causes the spreading of the ordinary Schrödinger wave packet; this spreading is compensated by the nonlinearity $F(\rho)$ and by the EIP contribution $-(m\,u^2/2)\,(1+\kappa\,\rho)^{-2}$. Therefore it is possible to build up a non-spreading solitary wave.
After the introduction of the function

$$y(\rho) = \left(\frac{1}{\rho}\frac{d\,\rho}{d\,\xi}\right)^2 \ , \tag{5.1.11}$$

so that

$$\frac{d\,y}{d\,\rho} = \frac{2}{\rho}\,\frac{d}{d\,\xi}\left(\frac{1}{\rho}\,\frac{d\,\rho}{d\,\xi}\right) \ , \tag{5.1.12}$$

Eq. (5.1.9) reduces to a first order linear differential equation:

$$\frac{d\,y}{d\,\rho} + \frac{y}{\rho} + \frac{(2\,m\,u/\hbar)^2}{\rho\,(1+\kappa\,\rho)^2} - \frac{8\,m}{\hbar^2}\frac{F(\rho)}{\rho} + \frac{8\,m\,\epsilon}{\hbar^2}\frac{1}{\rho} = 0 \ , \tag{5.1.13}$$

that can be easily integrated, giving:

$$y(\rho) = \frac{A}{\rho} - \frac{8\,m\,\epsilon}{\hbar^2} - \frac{(2\,m\,u/\hbar)^2}{1+\kappa\,\rho} + \frac{8\,m}{\hbar^2}\,\frac{\widetilde{U}(\rho)}{\rho} \ , \tag{5.1.14}$$

where the integration constant is $A-1/\kappa$ in order to obtain the right limit for $\kappa \to 0$. By comparing Eq. (5.1.11) to Eq. (5.1.14), we obtain:

$$\left(\frac{d\,\rho}{d\,\xi}\right)^2 = A\,\rho - \frac{8\,m\,\epsilon}{\hbar^2}\rho^2 - \left(\frac{2\,m\,u}{\hbar}\right)^2\,\frac{\rho^2}{1+\kappa\,\rho} + \frac{8\,m}{\hbar^2}\,\rho\,\widetilde{U}(\rho) \ . \tag{5.1.15}$$

The evaluation of the soliton shape is thus reduced to the solution of the first order ordinary differential equation (5.1.15). By introducing the dimensionless variables:

$$n = |\kappa|\,\rho \ , \tag{5.1.16}$$

and

$$\chi = \frac{2\,m\,u}{\hbar}\,(x \mp u\,t) \ , \tag{5.1.17}$$

Eq. (5.1.15) takes the form:

$$\left(\frac{d\,n}{d\,\chi}\right)^2 = \alpha\,n + \beta\,n^2 + \gamma\,n\,\widehat{U}(n) - \frac{n^2}{1+\sigma\,n}\ , \tag{5.1.18}$$

where $\alpha = A\,|\kappa|\,(\hbar/2\,m\,u)^2$ and $\beta = -2\,\varepsilon/m\,u^2$ are the new integration constants, $\gamma = 2\,|\kappa|/m\,u^2$ and $\widehat{U}(n) \equiv \widetilde{U}(\rho)$. The parameter $\sigma = \kappa/|\kappa|$ assumes the value $+1$ when the inclusion principle holds ($\kappa > 0$, $n \geq 0$). Accordingly, for the exclusion principle ($\kappa < 0$, $0 \leq n \leq 1$) we have $\sigma = -1$. Here we have made use of the scaling properties (3.2.28) which permit us to take into account only the two special case $\kappa = \pm 1$. We note, finally, that while solving the Eq. (5.1.18), we have to take into account the two arbitrary constants α and β that define a family of solutions.

To determine the soliton solutions we must search the solutions of the first order differential equation (5.1.18), varying the arbitrary constants α and β, while the sign $\sigma = \pm 1$ is kept fixed. Eq. (5.1.18) after integration gives:

$$\pm\,\chi = \int^{n} \left(\frac{\alpha\,n + (\sigma\,\alpha + \beta - 1)\,n^2 + \sigma\,\beta\,n^3 + \gamma\,n\,(1+\sigma\,n)\,\widehat{U}}{1+\sigma\,n}\right)^{-\frac{1}{2}} dn\ . \tag{5.1.19}$$

From Eq. (2.3.16), we have for the soliton case:

$$< \widehat{P} > = \pm M\,u\ , \tag{5.1.20}$$

where M is defined by:

$$M = m \int\limits_{-\infty}^{+\infty} \frac{\rho}{1+\kappa\,\rho}\,dx\ . \tag{5.1.21}$$

By using Eq. (2.3.18) we can write the soliton energy as:

$$E = \frac{< \widehat{P}^2 >}{2\,m} + \frac{\kappa}{2}\,m\,u^2 \int\limits_{-\infty}^{+\infty} \left[\left(\frac{\rho}{1+\kappa\,\rho}\right)^2 + \widetilde{U}(\rho)\right] dx\ , \tag{5.1.22}$$

with

$$< \widehat{P}^2 > \equiv \hbar^2 \int \left|\frac{d\,\psi}{d\,x}\right|^2 dx\ . \tag{5.1.23}$$

To evaluate $< \widehat{P}^2 > /2\,m$ we take into account (5.1.3):

$$\begin{aligned} \frac{< \widehat{P}^2 >}{2\,m} &= \frac{\hbar^2}{4} \int \frac{1}{\rho}\left(\frac{d\,\rho}{dx}\right)^2 dx + \int \left(\frac{d\,s}{d\,x}\right)^2 \rho\,dx \\ &= \frac{\hbar^2}{4} \int \frac{1}{\rho}\left(\frac{d\,\rho}{dx}\right)^2 dx + (m\,u)^2 \int \frac{\rho}{(1+\kappa\,\rho)^2}\,dx\ , \end{aligned} \tag{5.1.24}$$

using Eq. (5.1.9) we obtain:

$$\frac{<\widehat{P}^2>}{2\,m} = m\,u^2 \int\limits_{-\infty}^{+\infty} \frac{\rho}{(1+\kappa\,\rho)^2}\,dx - \int\limits_{-\infty}^{+\infty} \rho\,\frac{d\,\widetilde{U}(\rho)}{d\rho}\,dx + \epsilon\,N \ . \tag{5.1.25}$$

Considering that $u =< \widehat{P} > /M$ and Eq. (5.1.25), the energy (5.1.22) satisfies the following soliton energy-momentum dispersion relation:

$$\begin{aligned} E &= \frac{<\widehat{P}>^2}{2\,M}\left[1+\int\limits_{-\infty}^{+\infty}\frac{\rho}{(1+\kappa\,\rho)^2}\,dx\right]\left[\int\limits_{-\infty}^{+\infty}\frac{\rho}{1+\kappa\,\rho}\,dx\right]^{-1} \\ &+ \int\limits_{-\infty}^{+\infty}\left[\widetilde{U}(\rho)-\rho\,\frac{d\,\widetilde{U}(\rho)}{d\rho}\right]\,dx + \epsilon\,N \ . \end{aligned} \tag{5.1.26}$$

If we chose the constant ϵ, appearing in the phase of ψ, as:

$$\epsilon\,N = \int\limits_{-\infty}^{+\infty}\left[\rho\,\frac{d\,\widetilde{U}(\rho)}{d\rho}-\widetilde{U}(\rho)\right]\,dx - \frac{1}{2}\,m\,u^2\int\limits_{-\infty}^{+\infty}\frac{\rho}{(1+\kappa\,\rho)^2}\,dx \ , \tag{5.1.27}$$

the energy-momentum dispersion relation assumes the expression:

$$E = \frac{<\widehat{P}>^2}{2\,M} \ , \tag{5.1.28}$$

which is related to a free particle of mass M, traveling with momentum $<\widehat{P}>= \pm M\,u$.

Let us study Eq. (5.1.18) defining the shape of the solitons. We discuss the case $\widehat{U}(n) = 0$ and $\beta = 1$. As an example, we consider the case where the EIP is reduced to the inclusion principle (boson case) and the particle interaction is attractive ($\sigma = 1$). When $\alpha = 0$ we can obtain the expression of the soliton explicitly. In fact, Eq. (5.1.19) now becomes:

$$\pm\chi = \int^{n}\sqrt{\frac{1+n}{n^3}}\,dn \ , \tag{5.1.29}$$

and after integration we obtain the following implicit solution:

$$\text{arcoth}\sqrt{\frac{1+n}{n}} - \sqrt{\frac{1+n}{n}} = \pm\frac{1}{2}\chi \ . \tag{5.1.30}$$

The value $n(\chi)$, at the origin $n(0) = n_\circ$, is the solution of the transcendent equation:

$$\tanh\sqrt{\frac{1+n_\circ}{n_\circ}} = \sqrt{\frac{n_\circ}{1+n_\circ}} \;, \tag{5.1.31}$$

with $n_\circ = 2.27671$. We note that for $\chi \to \pm\infty$, the asymptotic form is $n \simeq 4/\chi^2$. A soliton with this behavior for $\chi \to \pm\infty$ was taken into account recently in Ref. [122]. We note also that at the origin the soliton has an angular point because its first kinetic is discontinuous:

$$\left(\frac{d\,n}{d\,\chi}\right)_{\chi=0} = \pm\sqrt{\frac{n_\circ^3}{n_\circ+1}} \;.$$

The wave function of the soliton is:

$$\psi(\chi,\, t) = \sqrt{\frac{n(\chi)}{\kappa}}\,\exp\left\{-i\left[\sqrt{\frac{1+n(\chi)}{n(\chi)}} - \sqrt{\frac{1+n_\circ}{n_\circ}} + \frac{\epsilon}{\hbar}\,t\right]\right\} \;, \tag{5.1.32}$$

where

$$\epsilon = -m\,u^2\,\frac{n_\circ^2}{(1+n_\circ)^2} \;, \tag{5.1.33}$$

obtained using Eq. (5.1.27), so that the soliton has a particle-like behavior. The expression (5.1.32) of the soliton can be used to calculate explicitly the quantities N, $< P >$ and E. Because of the Eqs. (5.1.27) and (5.1.33) the number N is:

$$N = \frac{\hbar}{\kappa\,m\,u}\,(1+n_\circ)\sqrt{\frac{1+n_\circ}{n_\circ}} \;. \tag{5.1.34}$$

On the other hand if we define the mass M as in Eq. (5.1.21):

$$M = \frac{2}{1+n_\circ}\,m\,N \;, \tag{5.1.35}$$

the momentum is given by $< P >= M\,u$ and the energy is $E =< P >^2 /2\,M$.

5.2 Effective potential

In this section we prove that the unitary nonlinear transformation (3.3.9) ($\psi(\xi,\, t) \to \phi(\xi,\, t)$) studied in section 3.3, when applied on the solitonic states $\psi(\xi,\, t)$ exists and

that the new states $\phi(\xi,\, t)$ are solutions of a Schrödinger equation with an algebraic real nonlinearity.

Let us consider the unitary transformation:

$$\psi(\xi,\, t) \to \phi(\xi,\, t) = \mathcal{U}(\xi)\, \psi(\xi,\, t) \ , \tag{5.2.36}$$

where $\mathcal{U}(\xi)$ is given by:

$$\mathcal{U}(\xi) = \exp\left\{\frac{i}{\hbar}\left[\pm m\, u\, \xi - s(\xi)\right]\right\} \ . \tag{5.2.37}$$

The new wave function $\phi(\xi,\, t)$ has the same amplitude of the wave function $\psi(\xi,\, t)$ but a different phase:

$$\phi(\xi,\, t) = \rho(\xi)^{1/2}\, \exp\left\{\frac{i}{\hbar}\left(\pm m\, u\, \xi - \epsilon\, t\right)\right\} \ . \tag{5.2.38}$$

The unitary transformation can be rewritten as:

$$\psi(\xi,\, t) = \exp\left[-\frac{i}{\hbar}\, \Gamma(\xi)\right]\, \phi(\xi,\, t) \ , \tag{5.2.39}$$

where

$$\Gamma(\xi) = \pm m\, u\, \left(\xi - \int\limits_0^{\xi} \frac{d\,\xi'}{1 + \kappa\, \rho(\xi')}\right) \ . \tag{5.2.40}$$

After deriving (5.2.39) we obtain the following relations:

$$\frac{\partial\, \psi}{\partial\, t} = \left[\frac{\partial\, \phi}{\partial\, t} - \frac{i}{\hbar}\frac{\partial\, \Gamma}{\partial\, t}\, \phi\right]\, e^{-i\, \Gamma/\hbar} \ , \tag{5.2.41}$$

$$\frac{\partial^2\, \psi}{\partial\, \xi^2} = \left[-\frac{i}{\hbar}\, \frac{\partial^2\, \Gamma}{\partial\, \xi^2}\, \phi - \frac{1}{\hbar^2}\, \left(\frac{\partial\, \Gamma}{\partial\, \xi}\right)^2\, \phi - i\, \frac{2}{\hbar}\frac{\partial\, \Gamma}{\partial\, \xi}\frac{\partial\, \phi}{\partial\, \xi} + \frac{\partial^2\, \phi}{\partial\, \xi^2}\right]\, e^{-i\, \Gamma/\hbar} \ . \tag{5.2.42}$$

By considering (5.1.2), (5.1.7) and (5.2.40) we obtain:

$$\frac{\partial\, S}{\partial\, \xi} = \frac{1}{\kappa\, \rho}\, \frac{\partial\, \Gamma}{\partial\, \xi} \ , \tag{5.2.43}$$

and then the current j, given by (2.1.14) becomes:

$$j = \frac{1}{\kappa\, m}\, (1 + \kappa\, \rho)\, \frac{\partial\, \Gamma}{\partial\, \xi} \ . \tag{5.2.44}$$

In the case of soliton states the Schrödinger equation (2.1.25) is:

$$
\begin{aligned}
i\hbar\,\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2\,m}\,\frac{\partial^2\psi}{\partial\xi^2}+F(\rho)\,\psi \\
&+ \kappa\,\frac{m}{\rho}\left(\frac{j}{1+\kappa\,\rho}\right)^2\psi-i\,\kappa\,\frac{\hbar}{2\,\rho}\,\frac{\partial}{\partial\xi}\left(\frac{j\,\rho}{1+\kappa\,\rho}\right)\psi\ .
\end{aligned}
\tag{5.2.45}
$$

By using (5.2.41), (5.2.42) and (5.2.44) we may write Eq (5.2.45) in the form:

$$
\begin{aligned}
\frac{\partial\Gamma}{\partial t}\,\phi+i\,\hbar\,\frac{\partial\phi}{\partial t} &= -\frac{\hbar^2}{2\,m}\,\frac{\partial^2\phi}{\partial\xi^2}+\frac{i\,\hbar}{2\,m}\,\frac{\partial^2\Gamma}{\partial\xi^2}\,\phi+\frac{2+\kappa\,\rho}{2\,m\,\kappa\,\rho}\left(\frac{\partial\Gamma}{\partial\xi}\right)^2\phi \\
&+\frac{i\,\hbar}{m}\,\frac{\partial\Gamma}{\partial\xi}\,\frac{\partial\phi}{\partial\xi}-\frac{i\,\hbar}{2\,m\,\rho}\,\frac{\partial}{\partial\xi}\left(\frac{\partial\Gamma}{\partial\xi}\,\rho\right)\phi+F(\rho)\,\phi\ .
\end{aligned}
\tag{5.2.46}
$$

We use now the relation $\partial\,\Gamma/\partial\,t=\mp u\,\partial\,\Gamma/\partial\,\xi$ with:

$$
\frac{\partial\Gamma}{\partial\xi}=\pm m\,u\,\frac{\kappa\,\rho}{1+\kappa\,\rho}
\tag{5.2.47}
$$

(easily derivable from (5.1.7) and (5.2.44)), Eq. (5.2.46) and:

$$
-\frac{i\,\hbar}{2\,m\,\rho}\,\frac{\partial\Gamma}{\partial\xi}\,\frac{\partial\rho}{\partial\xi}\,\phi+\frac{i\,\hbar}{m}\,\frac{\partial\Gamma}{\partial\xi}\,\frac{\partial\phi}{\partial\xi}=\frac{i\,\hbar}{2\,m}\left(\phi^*\,\frac{\partial\phi}{\partial\xi}-\phi\,\frac{\partial\phi^*}{\partial\xi}\right)\frac{\partial\Gamma}{\partial\xi}\,\frac{\phi}{\rho}\ .
\tag{5.2.48}
$$

If we take into account that $j_\phi=\pm u\,\rho$ we arrive to the following NLSE:

$$
i\,\hbar\,\frac{\partial\phi}{\partial t}=-\frac{\hbar^2}{2\,m}\,\frac{\partial^2\phi}{\partial\xi^2}+\frac{1}{2}\,m\,u^2\,\kappa\,\rho\,\frac{2+\kappa\,\rho}{(1+\kappa\,\rho)^2}\,\phi+F(\rho)\,\phi\ .
\tag{5.2.49}
$$

Let us now introduce the variable x and define $F_{\rm eff}(\rho)$:

$$
F_{\rm eff}(\rho)=F(\rho)+\frac{1}{2}\,m\,u^2\,\kappa\,\rho\,\frac{2+\kappa\,\rho}{(1+\kappa\,\rho)^2}\ ,
\tag{5.2.50}
$$

then Eq. (5.2.49) can be rewritten as:

$$
i\,\hbar\,\frac{\partial\phi}{\partial t}=-\frac{\hbar^2}{2\,m}\,\frac{\partial^2\phi}{\partial x^2}+F_{\rm eff}(\rho)\,\phi\ .
\tag{5.2.51}
$$

By means of $F_{\rm eff}(\rho)=d\,U_{\rm eff}(\rho)/d\,\rho$ we can introduce the potential $U_{\rm eff}(\rho)$ which is given by:

$$
U_{\rm eff}(\rho)=\widetilde{U}(\rho)+\frac{1}{2}\,m\,u^2\,\kappa\,\frac{\rho^2}{1+\kappa\,\rho}\ .
\tag{5.2.52}
$$

We remark that the term $(\kappa\, m\, u^2/2)\, \rho^2/(1+\kappa\, \rho)$ originates from the $U_{\rm EIP}(\rho,\, j)$ and represent the EIP effect on the shape of the soliton. We remember also that the transformation (5.2.36) is noncanonical. In the case of soliton solution (and only for this special solution) we are able to write a new evolution equation derivable by a variational principle, starting from the density Lagrangian:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_0 - U_{\rm eff} \ , \tag{5.2.53}$$

where $\mathcal{L}_0$ is the density of Lagrangian of the linear Schrödinger equation. Because of the noncanonicity of (5.2.36) $U_{\rm eff}$ is not obtainable directly from $U_{\rm EIP}$.

Let us consider the transformation recently introduced by Doebner and Goldin [54, 95]:

$$\psi(t,\, x) \to \phi(t,\, x) = \sqrt{\rho(t,\, x)}\, \exp\left[i\left(\frac{\gamma(t)}{2}\, \log\, \rho(t,\, x) + \frac{\lambda(t)}{\hbar}\, S(t,\, x) + \theta(t,\, x)\right)\right] \tag{5.2.54}$$

Transformation (5.2.54) defines a class of non-linear gauge transformations, varying the parameters $\gamma(t)$, $\lambda(t)$ and $\theta(t,\, x)$ and has the important property of making linear a particular sub-family of equations belonging to the Doebner-Goldin equation family. By comparing Eq. (5.2.37) to Eq. (5.2.54) we can note that the transformation introduced in this work is a particular case of the more general transformation introduced by Doebner and Goldin. Transformation (5.2.37) is limited to the soliton states without linearizing the Schrödinger equation describing these states. It reduces the complex nonlinearity of the Schrödinger equation to another real one. By following the same procedure of the previous section, we can determine the shape of the solitonic solutions of Eq. (5.2.51), that are the same states of Eq. (2.1.25) modulo the phase $S(\xi,\, t)$. Of course, standing on the unitarity of Eq. (5.2.37), the equation for the shape of the soliton is the same of Eq. (5.1.18). The problem of searching the solitonic solutions of Eq. (2.1.25) with derivative complex nonlinearities due to the EIP, is reduced now to the search of the solitonic solutions of a Schrödinger equation with analytic real nonlinearity.

We consider now the nonlinear potential:

$$\widetilde{U}(\rho) = \widetilde{U}_0(\rho) - \frac{1}{2}\, m\, u^2\, \kappa\, \frac{\rho^2}{1+\kappa\, \rho} \ , \tag{5.2.55}$$

where $\widetilde{U}_0(\rho)$ is again an analytic real arbitrary potential in ρ and the second term is selected with the scope of eliminating the effect of the EIP. Eq. (5.1.18) now takes the form:

$$\left(\frac{d\, n}{d\, \chi}\right)^2 = \alpha\, n - \frac{2\, \epsilon}{m\, u^2}\, n^2 + \frac{2\, |\kappa|}{m\, u^2}\, n\, \widehat{U}_0(n) \ . \tag{5.2.56}$$

Equation (5.2.56) is identical to the equation of the solitonic shape that we may find in literature if we take a NLSE with the analytic non-linear potential $\widetilde{U}_0(\rho)$ [121]. Then Eq. (5.2.56) allows us to use the soliton solutions of NLSEs available in literature.

5.3 Applications

As a first application of the results obtained in the previous section, we derive the nonlinear potential $\widetilde{U}(\rho)$ which, when is present simultaneously with the EIP potential $U_{\rm EIP}(\rho,\, \boldsymbol{j})$, permits the formulation of a soliton with shape given by: $\rho(\xi) \propto [\cosh(b\,\xi)]^{-2}$. We start considering the non-linearity [39, 123]:

$$U_0(\rho) = -\frac{\mu}{2}\,\rho^2\ . \tag{5.3.57}$$

The Schrödinger equation with this non-linearity has been recently used to study the Bose-Einstein condensation (BEC) [41, 42, 43, 44, 45]. In the Gross-Pitayevski equation the parameter μ is given by:

$$\mu = \frac{4\,\pi\,\hbar^2\,N\,a}{m}\ , \tag{5.3.58}$$

where N is the number of atoms in the condensate, m their mass and a is the s-wave triplet scattering length. Its value is assumed to range in the interval $85\,a_0 < a < 140\,a_0$, a_0 being the Bohr radius [124].
Set $\alpha' = 0$ and $\mu > 0$, Eq. (5.2.56) with the potential (5.3.57) is easily integrable obtaining:

$$\rho(\xi) = \frac{N\,b}{2}\ [\cosh(b\,\xi)]^{-2}\ , \tag{5.3.59}$$

where b is a dimensionless constant defined as:

$$b = \frac{\mu\,m\,N}{2\,\hbar^2}\ , \tag{5.3.60}$$

and the normalization $N = \int |\psi|^2\,d\xi$, that fixes the parameter $\epsilon = -\mu^2\,m\,N^2/8\,\hbar^2$, has been taken into account.
The phase $S(\xi,\,t)$ of the soliton takes the form:

$$S(\xi,\,t) = -\epsilon\,t \pm m\,u\,\xi \mp m\,u\,\frac{c}{b}\ \tanh^{-1}\,[c\,\tanh(b\,\xi)]\ , \tag{5.3.61}$$

with:

$$c = \left(1 + \frac{2}{\kappa\,b\,N}\right)^{-\frac{1}{2}}\ , \tag{5.3.62}$$

a dimensionless constant.
The EIP effect modifies the phase of the soliton. In the limit $\kappa \to 0$, i.e. when the EIP is switched off, the phase of the soliton becomes equal to the phase of the soliton of the cubic Schrödinger equation.

Finally, we remark that in the case of a pure exclusion principle ($\kappa < 0$), the soliton exists, as we can see from (5.3.61), only if:

$$4\,\hbar^2 > |\kappa|\,\mu\,m\,N^2 \;. \tag{5.3.63}$$

If we take into account the maximum value of the quantity $\rho(\xi)$, that is $\rho(0) = \mu\,m\,N^2/4\,\hbar^2$ and the maximum number of particles that can be put in a site:

$$\rho_{\max} = \frac{1}{|\kappa|} \;, \tag{5.3.64}$$

Eq. (5.3.63) can be written in the form:

$$\rho(0) < \rho_{\max} \;. \tag{5.3.65}$$

This imposes no violation of the exclusion principle in the central site, where the maximal occupation exists and, consequently, no violation of the exclusion principle on all the other points of the space.
Taking into account Eqs. (5.2.55) and (5.3.57), we can write the potential $\widetilde{U}(\rho)$, which generates the soliton given by Eqs. (5.3.59) and (5.3.61), as:

$$\widetilde{U}(\rho) = -\frac{\mu}{2}\,\rho^2 - \kappa\,\frac{1}{2}\,m\,u^2\frac{\rho^2}{1+\kappa\,\rho} \;. \tag{5.3.66}$$

Chapter VI

EIP-Gauged Schrödinger Model: Chern-Simon Vortices

In this chapter we describe a possible application of the model introduced in the chapter IV, describing systems of interacting particles obeying to EIP. Of course, the model can be used whenever the physical circumstances are such that collective excitations of charge particles occur like, for instance, in the study of degenerate plasmas. Here we study in some detail the properties of static, self-dual, Chern-Simons (CS) vortices. It was emphasized by several authors that CS theories could describe effects observed in the recently discovered high-T_c superconductors.
Before discussing the EIP-vortex solutions, it might be worthwhile to mention how such solutions were historically discovered.
One of the most discussed topics in condensed matter is undoubtedly the superconductivity phenomenon. In the original Ginzburg-Landau model [99], the order parameter described by the field ψ interacts with a Maxwell like gauge field. Although the equations depend on three free parameters, two of them can be eliminated by an appropriated scaling, leaving only one relevant physical parameter λ. Superconductors of type I and II correspond respectively to the value of the parameter λ less or greater than one. Subsequently it was shown that the Ginzburg-Landau model admits vortex like solutions [125]: localized flux tube surrounded by a circulating supercurrent. These vortices were experimentally observed in superconductors of type II. In the Bogomol'nyi limit, $\lambda = 1$ vortex like solutions acquire interesting properties, in particular static solutions are admitted because of the absence of forces exchanged among vortices [126].
Nielsen and Olesen [127] rediscovered these solutions in the context of the relativistic generalization of the Ginzburg-Laudau model, known as the Abelian Higgs model [128, 129, 130]. These authors were looking for string-like objects in relativistic field

theory. It turns out that these vortices have finite energy per unit length in 3+1 dimensions (i.e. finite energy in 2+1 dimensions as the vortex dynamics is essentially confined to the $x-y$ plane) quantized flux, but are electrically neutral and have zero angular momentum. In particle physics theories these solutions may be interpreted as strings joining confined quarks, while in cosmology theories may be interpreted as cosmic strings produced at a phase transition in the early history of our universe [131].

Subsequently, Julia and Zee [132] showed that the $SO(3)$ Georgi-Glashow model, which admits t'Hooft-Polyakov monopole solutions, also admits its charged generalization named dyon solutions with finite energy and finite, non zero, electric charge. It was then natural for them to enquire whether the Abelian Higgs model, which admits neutral vortex solutions with finite energy (in 2+1 dimensions), also admits its charged generalization or not. In the same paper, Julia and Zee discussed this question and showed that the answer is negative, i.e. unlike the monopole case, the Abelian Higgs model does not admit charged vortices with finite energy and finite and non zero electric charge. More than ten years later, Paul and Khare [133] showed that the Julia-Zee negative result can be overcome if one adds the CS term to the Abelian Higgs model. In particular, was showed that the Abelian Higgs model with CS term in 2+1 dimensions admits charged vortex solutions of finite energy and quantized, finite, Nöther charge as well as flux. As an extra bonus, it was found that these vortices also have non zero, finite angular momentum that is in general fractional. This strongly suggested that these charged vortices could in fact be charged anyons, as it was rigorously shown by Fröhlich and Machetti [134].

In the last years, Jackiw and Pi [115], studying the nonrelativistic reduction of the Abelian Higgs-CS model to the Schrödinger-CS model, where a nonlinear potential $U(\rho) \propto \rho^2$ appears, were able to resolve it in the self-dual limit. In particular, this model admits nontopological vortex solutions in which the electric charge, the magnetic flux and the angular momentum are altogether quantized quantities, while energy and momentum are equal to zero, a feature of the self-dual solutions. The same model was studied in presence of an external magnetic field [135], the importance of it being evident if we take into account the applications of the CS theory to the fractional quantum Hall effect [103].

Also the CS-Abelian Higgs model admits self-dual solutions when the nonlinear potential takes the form: $U(\rho) \propto \rho\,(\rho - v^2)^2$. This potential introduces a spontaneous symmetry-breaking mechanism and consequently the topological vortex makes its appearance in the symmetry-breaking phase. The solutions of the relativistic model may be considered as the analogous in (2+1) dimensions of the magnetic monopoles of 'tHooft-Polyakov [51, 128, 136].

Self-dual planar solitons can be also found in many theories of fermions where the

dynamics is described both in the frame of the Dirac equation with gauge field interacting by means of CS [137] or Maxwell-CS terms [138], and in the nonrelativistic model of the Lévy-Leblond-CS equation [139]. In these theories, as in the scalar ones, after a calibration of the nonlinear potential, it is possible to show that each component of the spinorial field satisfies the Bogomol'nyi equation reducible to the Liouville differential equation whose solutions are known.
An interesting generalization of the Abelian Higgs-CS model was formulated by Torres [140] and extended by other authors [141, 142, 143]. In this model a matter field is coupled in a non minimal way with a gauge field through the presence of a term that behaves as an anomalous magnetic moment. In the Bogomol'nyi limit the model admits topological and nontopological vortex solutions with non quantized electric charge, magnetic flux and angular momentum while in the topological sector the energy is quantized and infinitely degenerate.
For an extensive review on CS theories see Ref. [144].

6.1 Static solutions

Let us describe the model studied in this chapter. We consider the particular situation in absence of the external potential $V(t,\,\boldsymbol{x})$ and when the interaction of the gauge field is described exclusively by means of the CS term. This approximation is specially relevant in the context of condensed matter systems since in the long wave length limit, (low energy domain), the high-derivative Maxwell terms are dominated by the first order derivative CS one [51, 115, 136]. Thus we set $\gamma=0$ and $V=0$ so that the motion equations for the matter field obtained from (4.1.3) and (4.1.5) become:

$$i\,\hbar\,c\,D_0\,\psi=-\frac{\hbar^2}{2\,m}\boldsymbol{D}^2\psi+\Lambda(\rho,\,\boldsymbol{J})\,\psi+F(\rho)\,\psi\ , \tag{6.1.1}$$

with $F(\rho)=d\,\widetilde{U}(\rho)/\rho$ and the nonlinear term $\Lambda(\rho,\,\boldsymbol{J})$ given by Eq. (4.1.15). As we will show, the arbitrary nonlinear potential $U(\rho)$ can be selected in order to permit the existence of self-dual vortex solutions for Eq. (6.1.1). In the self-dual limit we are able to decouple the gauge fields equations from the matter ones and reduce the evolution equation of the ψ field to an ordinary differential equation which will be solved numerically by means of the Runge-Kutta algorithm.
The motion equations for the gauge fields, when only the CS term is present, become:

$$\frac{g}{2}\,\varepsilon^{\nu\rho\mu}\,F_{\rho\mu}=\frac{e}{c}\,J^\nu\ . \tag{6.1.2}$$

In section 4.3 we have seen that Eqs. (6.1.2) can be solved allowing us to write the gauge fields as a function of the sources ρ and $\boldsymbol{J}$. Therefore Eq. (6.1.1) can be seen as an highly nonlinear Schrödinger equation for the only field ψ. The model that we come to study is a continuous deformation, in the parameter κ, of the model presented and studied in Ref. [51] by Jackiw and Pi.

To search the static solutions of Eqs. (6.1.1) and (6.1.2) we make use of the property of invariance under gauge transformation of the Lagrangian:

$$\psi \rightarrow e^{-i(e/\hbar\, c)\,\omega}\psi \ , \tag{6.1.3}$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \omega \ . \tag{6.1.4}$$

We choose to work in the London gauge where the matter field ψ becomes real, because its phase $Arg(\psi)$ is absorbed by the field A_μ. By using Eqs. (6.1.3) and (6.1.4), and setting the parameter $\omega = (\hbar\, c/e)\, Arg(\psi)$, we can see that after the introduction of the new fields:

$$\phi(\boldsymbol{x}) = |\psi(\boldsymbol{x})| \ , \tag{6.1.5}$$

$$\chi_\mu(\boldsymbol{x}) = A_\mu(\boldsymbol{x}) + \frac{\hbar\, c}{e}\partial_\mu\, Arg\left[\psi(\boldsymbol{x})\right] \ , \tag{6.1.6}$$

the expression of the Hamiltonian becomes:

$$H = \int \left[\frac{\hbar^2}{8\, m}\,\frac{(\boldsymbol{\nabla}\, \rho)^2}{\rho} + \frac{e^2}{2\, m\, c^2}\,\rho\,(1+\kappa\,\rho)\,\boldsymbol{\chi}^2 + \widetilde{U}(\rho)\right] d^2x \ , \tag{6.1.7}$$

where $\rho = \phi^2$ is the particle density. We observe that for $\kappa < 0$ we have $0 \leq \rho \leq 1/|\kappa|$ while for $\kappa > 0$ we have $\rho > 0$ so that $1 + \kappa\,\rho \geq 0$. Therefore, from (6.1.7) we see that, if $\widetilde{U}(\rho) = 0$, H is a semidefinite positive quantity limited below by zero. If we take into account the relation:

$$\begin{aligned} &\left|\partial_\pm \log\rho + i\,\frac{2\,e}{\hbar\, c}\,(1+\kappa\,\rho)^{1/2}\,\chi_\pm\right|^2 = \left(\frac{\boldsymbol{\nabla}\rho}{\rho}\right)^2 + \left(\frac{2\,e}{\hbar\, c}\right)^2\,(1+\kappa\,\rho)\,\boldsymbol{\chi}^2 \\ &+ i\frac{2\,e}{\hbar\, c}\,(1+\kappa\,\rho)^{1/2}\,(\chi_\pm\,\partial_\mp \log\rho - \chi_\mp\,\partial_\pm \log\rho) \ , \end{aligned} \tag{6.1.8}$$

with $\partial_\pm = \partial_1 \pm i\,\partial_2$ and $\chi_\pm = \chi_1 \pm i\,\chi_2$, the Hamiltonian becomes:

$$\begin{aligned} H &= \frac{\hbar^2}{8\, m}\int \left|\partial_\pm \log\rho + i\,\frac{2\,e}{\hbar\, c}\,(1+\kappa\,\rho)^{1/2}\,\chi_\pm\right|^2 \rho\, d^2x \\ &\mp \frac{e\,\hbar}{3\, m\, c\, \kappa}\,\epsilon^{ij}\int \chi_i\,\partial_j\left[(1+\kappa\,\rho)^{3/2} - 1\right]\, d^2x + \int \widetilde{U}(\rho)\, d^2x \ . \end{aligned} \tag{6.1.9}$$

The second term in Eq. (6.1.9) is obtained by taking into account the identity:

$$(1+\kappa\,\rho)^{1/2}\,\partial_i\rho=\frac{2}{3\,\kappa}\,\partial_i\,\left[(1+\kappa\,\rho)^{3/2}-1\right]\ . \tag{6.1.10}$$

This term can be integrated by part. If we neglect the surface terms and take into account the Gauss law given by the time component of Eq. (6.1.2):

$$-g\,B=e\,\rho\ , \tag{6.1.11}$$

the Hamiltonian assumes the form:

$$\begin{aligned} H &= \frac{\hbar^2}{8\,m}\int\left|\partial_\pm\log\rho+i\,\frac{2\,e}{\hbar\,c}\,(1+\kappa\,\rho)^{1/2}\,\chi_\pm\right|^2\rho\,d^2x \\ &\quad+\int\left\{\widetilde{U}(\rho)\mp\frac{e^2\,\hbar}{3\,m\,c\,g\,\kappa}\left[(1+\kappa\,\rho)^{3/2}-1\right]\,\rho\right\}\,d^2x\ . \end{aligned} \tag{6.1.12}$$

We introduce now the quantities:

$$\mathcal{H}_0 = \frac{\hbar^2}{8\,m}\left|\partial_\pm\log\rho+i\,\frac{2\,e}{\hbar\,c}\,(1+\kappa\,\rho)^{1/2}\,\chi_\pm\right|^2\rho\ , \tag{6.1.13}$$

$$K(\rho) = \frac{1}{\kappa}\,\left[(1+\kappa\,\rho)^{3/2}-1\right]\,\rho\ , \tag{6.1.14}$$

$$\alpha = \mp\frac{e^2}{\hbar\,c\,g}\ , \tag{6.1.15}$$

and observe that $K(\rho)\geq 0$, $\forall\,\kappa\in I\!R$ and $\rho\geq 0$. The Hamiltonian assumes the form:

$$H=\int\mathcal{H}_0\,d^2x+\int\left[\widetilde{U}(\rho)+\frac{\hbar^2\,\alpha}{3\,m}\,K(\rho)\right]\,d^2x\ . \tag{6.1.16}$$

We observe that, independently on the sign of α (which will be discuss below), the quantity $\mathcal{H}_0+(\hbar^2\,\alpha/3\,m)\,K(\rho)\geq 0$.
Due to the canonicity of the system, the motion equation (6.1.1) in the Hamilton formalism and in the case of static solutions becomes:

$$\frac{\delta\,H}{\delta\,\psi^*}=0\ , \tag{6.1.17}$$

which can be written as:

$$\frac{\delta}{\delta\,\psi^*}\int\mathcal{H}_0\,d^2x+\frac{\partial}{\partial\,\rho}\left[\widetilde{U}(\rho)+\frac{\hbar^2\,\alpha}{3\,m}\,K(\rho)\right]\,\psi=0\ . \tag{6.1.18}$$

Following Ref. [51, 115] in which the Jackiw and Pi model ($\kappa = 0$) is discussed, it is easy to realize that the relations:

$$\partial_\pm \log \rho + i \, \frac{2\,e}{\hbar\, c}\, (1 + \kappa\, \rho)^{1/2}\, \chi_\pm = 0 \; . \tag{6.1.19}$$

$$U(\rho) + \frac{\alpha\, \hbar^2}{3\, m}\, K(\rho) = 0 \; , \tag{6.1.20}$$

ensure that the energy of the system vanishes. Eqs. (6.1.19) are two differential equations of the first order that permit to write the gauge field χ_+ and χ_- as functions of the field ρ, while the (6.1.20) fixes the analytical expression of the nonlinear potential $\widetilde{U}(\rho)$. In the limit $\kappa \to 0$, Eq. (6.1.19) reduces to the Bogomol'nyi equation [126], while in the same limit the potential $\widetilde{U}(\rho)$ given by (6.1.20) becomes proportional to ρ^2, in agreement with the results of Ref. [51, 115]. Making use of Eqs. (6.1.19) we can determine the field $\boldsymbol{\chi}$ if the expression of the field ρ is known:

$$\boldsymbol{\chi} = \pm \frac{\hbar\, c}{2\, e} \frac{\boldsymbol{\nabla} \wedge \log \rho}{(1 + \kappa\, \rho)^{1/2}} \; . \tag{6.1.21}$$

If we take into account the expression of the current (4.1.11), in the London gauge:

$$\boldsymbol{J} = - \frac{e}{m\, c}\, \rho\, (1 + \kappa\, \rho)\, \boldsymbol{\chi} \; , \tag{6.1.22}$$

and using Eq. (6.1.21), we can write $\boldsymbol{J}$ as a function of the field ρ:

$$\boldsymbol{J} = \mp \frac{\hbar}{2\, m}\, (1 + \kappa\, \rho)^{1/2}\, \boldsymbol{\nabla} \wedge \rho \; . \tag{6.1.23}$$

It is easy to verify that $\boldsymbol{J}$ can be rewritten as a curl and therefore it is a fully transverse current. Finally, after integration of the relation:

$$\partial_i\, A^0 = - E^i = - \frac{e}{c\, g}\, \epsilon_{ij}\, J^j \; , \tag{6.1.24}$$

we obtain:

$$A_{\rm o} \equiv \chi_{\rm o} = \frac{\hbar^2\, \alpha}{3\, m\, e}\, \frac{1}{\rho}\, K(\rho) \; . \tag{6.1.25}$$

Equations (6.1.21) and (6.1.25) allow us to obtain the gauge field χ_μ when the shape of the matter field ρ is known. In the limit $\kappa \to 0$, the field χ_μ assumes the expression given in Ref. [51, 115], so we conclude that the model described by the Lagrangians (4.1.3) and (4.1.5) generalizes the Jackiw and Pi model when the system obeys to EIP.

Now we calculate the field ρ. Taking into account the relation $B = \boldsymbol{\nabla} \wedge \boldsymbol{\chi}$, from (6.1.21) it follows that:

$$B = \pm \frac{\hbar\, c}{2\, e} \boldsymbol{\nabla} \wedge \left[\frac{\boldsymbol{\nabla} \wedge \log \rho}{(1 + \kappa\, \rho)^{1/2}} \right] \ . \tag{6.1.26}$$

By using the Gauss law (6.1.11), we obtain the following second order differential equation for the field ρ:

$$\Delta\ \log \left[\frac{4}{\kappa}\ \frac{(1 + \kappa\, \rho)^{1/2} - 1}{(1 + \kappa\, \rho)^{1/2} + 1} \right] = -2\, \alpha\, \rho\ , \tag{6.1.27}$$

which, in the limit of $\kappa \rightarrow 0$, reduces to the Liouville differential equation: $\Delta\ \log \rho = -2\, \alpha\, \rho$. We are not able to find the analytical solutions of Eq. (6.1.27). Numerical radially symmetrical solutions for the fields ρ will be discussed in section 6.3. Here we only anticipate that, how in the limit case $\kappa = 0$, nonsingular, nonnegative solutions will be obtained when the numerical constant α is positive. Hence the $\mp$ sign must be chosen according that of g.

We consider now Eq. (6.1.1) in the London gauge for the static configurations:

$$\frac{\hbar^2}{2\, m} \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = \frac{e^2}{2\, m\, c^2} (1 + 2\, \kappa\, \rho)\, \boldsymbol{\chi}^2 + e\, \chi_0 + F(\rho)\ . \tag{6.1.28}$$

It is easy to verify that the fields $\boldsymbol{\chi}$, χ_0 and ρ given by Eqs. (6.1.21), (6.1.25) and (6.1.27), are the required solutions.

We remark that the self-dual character of Jackiw and Pi static solutions is not altered by the presence of EIP. This self-duality can be easily recognized if we define the following transformation:

$$\psi \rightarrow \eta = R(r)^{1/2}\, e^{-i\,(e/\hbar\, c)\, S}\ , \tag{6.1.29}$$

that changes the amplitude of the field ψ in:

$$\rho(r) \rightarrow R(r) = \frac{4}{\kappa}\ \frac{(1 + \kappa\, \rho)^{1/2} - 1}{(1 + \kappa\, \rho)^{1/2} + 1}\ . \tag{6.1.30}$$

Now Eq. (6.1.19) in the η field becomes:

$$\boldsymbol{D}_\pm\, \eta = 0\ . \tag{6.1.31}$$

As a consequence, the solutions of Eq. (6.1.19) are the ground states of the system. This can be easily demonstrated by computing the component of the energy-momentum tensor. We have previously noted that the energy vanishes for the

solutions of Eq. (6.1.19) well-behaved at infinity. The density of momentum T^{0i} obtained in section 4.3, can be written by using (6.1.19) as:

$$T^{0i} = \mp \frac{c\,\hbar}{\kappa}\,\epsilon^{ij}\partial_j\left[(1+\kappa\,\rho)^{1/2}-1\right] \ , \tag{6.1.32}$$

that appears to be transverse. In the same way, we can show that the flux density of energy and of linear momentum are:

$$\begin{aligned} T^{i0} &= \frac{\epsilon^{ij}}{2\,g}\left(\frac{e\,\hbar}{3\,m\,c\,k}\right)^2 \partial_j\left[(1+\kappa\,\rho)^3 - 2\,(1+\kappa\,\rho)^{3/2}+1\right] \ , & (6.1.33)\\ T^{ij} &= 0 \ , & (6.1.34)\end{aligned}$$

and (6.1.33) appears to be a transverse quantity. Then, apart from total derivative terms, the energy-momentum tensor vanishes for the solutions of Eq. (6.1.19).
Finally, the expression of the density of angular momentum is given by:

$$\epsilon_{ij}\,x^i\,T^{0j} = \pm\frac{\hbar\,c}{\kappa}\,\boldsymbol{\nabla}\cdot\left\{[(1+\kappa\,\rho)^{1/2}-1]\,\boldsymbol{x}\right\} \mp \frac{2\,\hbar\,c}{\kappa}\left[(1+\kappa\,\rho)^{1/2}-1\right] \ , \tag{6.1.35}$$

and, if the field ρ vanishes at infinity more rapidly than $1/x^2$, the expression of the angular momentum becomes:

$$< M >= \mp\frac{2\,\hbar}{\kappa}\int[(1+\kappa\,\rho)^{1/2}-1]\,d^2x \ . \tag{6.1.36}$$

An alternative expression, useful in the next section, can be obtained starting from Eq. (4.3.40) that we rewrite for convenience:

$$< M >= -\frac{i\,\hbar}{2}\int\boldsymbol{x}\wedge(\psi^*\,\boldsymbol{\nabla}\psi - \psi\,\boldsymbol{\nabla}\psi^*)\,d^2x - \frac{e}{c}\int\boldsymbol{x}\wedge\boldsymbol{A}\,\rho\,d^2x \ , \tag{6.1.37}$$

that, in the London gauge, assumes the form:

$$< M >= -\frac{e}{c}\int(\boldsymbol{x}\wedge\boldsymbol{\chi})\,\rho\,d^2x \ . \tag{6.1.38}$$

Using Eq. (6.1.11) it becomes:

$$< M >= \frac{g}{c}\int(\boldsymbol{x}\wedge\boldsymbol{\chi})\cdot(\boldsymbol{\nabla}\wedge\boldsymbol{\chi})\,d^2x \ , \tag{6.1.39}$$

which, with easy algebraic computation and taking into account the transversally of the field $\boldsymbol{\chi}$, can be written as:

$$< M >= \frac{g}{c}\oint\left[\frac{1}{2}\,(\boldsymbol{\chi}^2)\,\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{\chi})\,\boldsymbol{\chi}\right]\wedge d\boldsymbol{l} \ , \tag{6.1.40}$$

where the integral is to be taken both around a circle at infinity and around infinitesimal contours surrounding the poles of $\boldsymbol{\chi}$ (zeros of ρ).

6.2 Vortex like solutions

We study the solutions with angular symmetry (vortices) of the system. The wave function ψ for a vortex takes the form:

$$\psi(r,\,\theta) = \rho(r)^{1/2}\,\exp\left[-\frac{i\,e}{\hbar\,c}\,S(r,\,\theta)\right]\;, \tag{6.2.1}$$

where $\rho(r)$ satisfies the equation (6.1.27) that in polar coordinates becomes:

$$\frac{d}{d\,r}\left[r\,\frac{d}{d\,r}\log\frac{4}{\kappa}\,\frac{(1+\kappa\,\rho)^{1/2}-1}{(1+\kappa\,\rho)^{1/2}+1}\right] = -2\alpha\,\rho\,r\;. \tag{6.2.2}$$

Analytical solutions of (6.2.2) are not known, however in the limit of $\kappa\,\rho \to 0$ Eq. (6.2.2) becomes the Liouville differential equation whose solutions are known [145]. In section 6.1 we have shown that the value of the main physical observable associated to the solutions of Eq. (6.1.19) is zero. This statement is true with the exception of the mass, of the angular momentum and of the electric charge that, for the vortex solutions, is given by:

$$Q = 2\,\pi\,e\int\limits_0^\infty \rho(r)\,r\,dr\;. \tag{6.2.3}$$

Using Eq. (6.2.2), we can write:

$$Q = \frac{\pi\,e}{\alpha}\left[\lim_{r\to 0}\,r\,f(\kappa\,\rho) - \lim_{r\to\infty}\,r\,f(\kappa\,\rho)\right]\;, \tag{6.2.4}$$

with:

$$f(\kappa\,\rho) = \frac{1}{(1+\kappa\,\rho)^{1/2}}\frac{d}{d\,r}\log(\kappa\,\rho)\;, \tag{6.2.5}$$

where only the asymptotic behavior of ρ is invoked.

We remark that finite energy solutions of the system described by Hamiltonian (6.1.7) with the nonlinear potential given by (6.1.20) must vanish to infinity, while around the origin become a constant which can also be zero. We can expand the solution of Eq. (6.2.2) in correspondence of the zeros of ρ in a power series of ρ and, taking only the first order terms, we can see that the following equation $\rho'' - (\rho')^2/\rho + \rho'/r = 0$, where the prime indicates a derivative with respect to the r variable, must be satisfied. This equation admits power like solutions, so that we can write the following asymptotic behaviors of the solutions of Eq. (6.1.27):

$$\rho = C_0\,r^\beta\;, \qquad\qquad \beta > 0\;, \qquad\qquad r \to 0\;, \tag{6.2.6}$$

$$\rho = C_\infty \, r^{-\gamma} \; , \qquad\qquad \gamma > 0 \; , \qquad\qquad r \to \infty \; , \tag{6.2.7}$$

where C_0, C_∞, β and γ are integration constants.
Inserting (6.2.6) and (6.2.7) into Eq. (6.2.4) we obtain the expression of the charge:

$$Q = \frac{\pi\, e}{\alpha}\,(\beta + \gamma) \; , \tag{6.2.8}$$

holding for solutions vanishing at the origin.
Differently, we note from (6.2.4) that if the solutions $\rho(r)$ at the origin are equal to a constant, only their behavior at the infinity determines the electric charge. In this case the expression of the charge Q becomes:

$$Q = \frac{\pi\, e}{\alpha}\,\gamma \; . \tag{6.2.9}$$

By comparing Eqs. (6.2.8) and (6.2.9) we can conclude that the expression of the charge Q is given by (6.2.8) with $\gamma > 0$ and $\beta \geq 0$. Taking into account Eq. (6.1.21), we can deduce the asymptotic behavior for $r \to \infty$ of the gauge field χ^i:

$$\chi^i \underset{r\to\infty}{\sim} \pm \frac{\hbar\, c}{2\, e}\,\gamma\, \frac{\epsilon^{ij}\, x_j}{r^2} \; , \tag{6.2.10}$$

and

$$\chi^i \underset{r\to 0}{\sim} \mp \frac{\hbar\, c}{2\, e}\,\beta\, \frac{\epsilon^{ij}\, x_j}{r^2} \; , \tag{6.2.11}$$

for $r \to 0$ when $\rho \to 0$. The singularity of χ^i for $r \to 0$ is due to the particular choice of the gauge and can be eliminated by making a gauge transformation $U = \exp(i\, e\, \omega/\hbar\, c)$ with:

$$\omega = \mp \frac{\hbar\, c}{2\, e}\,\beta\,\theta \quad ; \qquad \theta = \arctan(x^2/x^1) \; . \tag{6.2.12}$$

By using the (6.2.6) and (6.2.12) we can write the expression of the field ψ for small values of r as:

$$\psi(r,\,\theta) = C_0\, r^\beta\, e^{\pm i\,\frac{\beta}{2}\,\theta} \; . \tag{6.2.13}$$

In order to derive the field ψ as a monodrome function, the positive number β should be an even integer:

$$\beta = 2\,(\nu - 1) \; , \tag{6.2.14}$$

where $\nu \in I\!N$ is the vorticity of the system. Therefore, we can write the charge Q in the form:

$$Q = \frac{\pi\, e}{\alpha}\,[2\,\nu + \gamma - 2] \; . \tag{6.2.15}$$

In section 6.2 we have remarked that when $\kappa \to 0$ the model studied in this paper reduces to the model of Jackiw and Pi. Therefore, in this limit, we have $\gamma \to 2\,(\nu+1)$ and, consequently, the discretization of the charge $Q = 4\,\pi\, e\, \nu/\alpha$ [51, 115]. This observation enables us to write γ in the form $\gamma = 2(\xi_{\nu,\kappa}+1)$ with $\xi_{\nu,\kappa} = \nu$ when $\kappa \to 0$. Therefore the electric charge Q can be written as:

$$Q = \frac{2\,\pi\, e}{\alpha}\,(\nu + \xi_{\nu,\kappa}) \; . \tag{6.2.16}$$

In the case $\kappa \neq 0$ the parameter $\xi_{\nu,\kappa}$ is a continuous function of the boundary conditions. As a consequence the charge Q loses its discretization. The behavior of the function $\xi_{1,\kappa}$ for a system with vorticity $\nu = 1$ will be studied in the next section. Finally, an alternative expression of the charge Q can be obtained from the integral form of Eq. (6.1.11):

$$Q = -g \int \boldsymbol{\nabla} \wedge \boldsymbol{\chi}\, d^2x = -g \oint \boldsymbol{\chi} \cdot d\boldsymbol{l} \; , \tag{6.2.17}$$

where the integral is performed both on the boundary at infinity and on the infinitesimal circles around the poles of the field $\boldsymbol{\chi}$ (zeros of ρ). By using (6.2.10) and (6.2.11) we obtain again the (6.2.8) and (6.2.9). This result means that the poles of the field $\boldsymbol{\chi}$ (zeros of ρ) are placed at the origin and at the infinity.
Finally, from Eq. (6.1.40), taking into account the asymptotic behavior of $\boldsymbol{\chi}$ and using (6.2.9) we obtain the angular momentum for the 1-vortex solutions:

$$< M >= \frac{Q^2}{4\,\pi\, c\, g} \; . \tag{6.2.18}$$

Analogously by using (6.2.15) we obtain the following expression of $< M >$ for the ν-vortex solution:

$$< M >= \frac{e\, Q\,(\xi_{\nu,\kappa}+1)}{c\, g\, \alpha} - \frac{Q^2}{4\,\pi\, c\, g} \; . \tag{6.2.19}$$

We observe that Eq. (6.2.19) in the case $\nu = 1$ reproduces exactly the (6.2.18) and then it holds for $\forall\, \nu \in I\!N$. We recall now that the spin of the ν-vortex solutions is given by $< S >= -Q^2/4\pi\, c\, g$. From (6.2.19) we have immediately the following expression for the orbital angular momentum $< L >=< M > - < S >$:

$$< L >= \frac{e\, Q\,(\xi_{\nu,\kappa}+1)}{c\, g\, \alpha} \; . \tag{6.2.20}$$

6.3 Numerical analysis

In section 3.2, we have shown that the transformation:

$$\begin{aligned}\kappa &\rightarrow \lambda^{-2}\,\kappa \ ,\\ \psi(t,\,\boldsymbol{x}) &\rightarrow \lambda\,\psi(t,\,\boldsymbol{x}) \ ,\end{aligned} \tag{6.3.1}$$

can be used to rescale, in the evolution equation of the system, the value of κ to ± 1. Introducing now the adimensional variables:

$$y = \sqrt{\frac{2\,\alpha}{|\kappa|}}\,r \ ,$$

$$n = |\kappa|\,\rho \ , \tag{6.3.2}$$

Eq. (6.2.2) becomes:

$$\frac{d}{d\,y}\left\{y\,\frac{d}{d\,y}\log\frac{[1+\sigma\,n(y)]^{1/2}-1}{[1+\sigma\,n(y)]^{1/2}+1}\right\} = -n(y)\,y \ , \tag{6.3.3}$$

where $\sigma = \kappa/|\kappa|$ takes the value $+1$ when the inclusion principle holds ($\kappa > 0$, $n \geq 0$) and, analogously for the exclusion principle ($\kappa < 0$, $0 \leq n \leq 1$), we have $\sigma = -1$. If we take into account Eq. (6.3.2), the expression for the matter field becomes:

$$\rho_\kappa(r) = \frac{1}{|\kappa|}\,n\left(\sqrt{\frac{2\,\alpha}{|\kappa|}}\,r\right) \ . \tag{6.3.4}$$

In order to integrate numerically Eq. (6.3.3), let us introduce the auxiliary field $z(y)$:

$$z(y) = y\,\frac{d}{d\,y}\log\frac{[1+\sigma\,n(y)]^{1/2}-1}{[1+\sigma\,n(y)]^{1/2}+1} \ , \tag{6.3.5}$$

so that the second order differential equation (6.3.3) can be transformed into a system of two first order differential equations:

$$\frac{d\,n(y)}{d\,y} = n(y)\,[1+\sigma\,n(y)]^{1/2}\,\frac{z(y)}{y} \ , \tag{6.3.6}$$

$$\frac{d\,z(y)}{d\,y} = -y\,n(y) \ . \tag{6.3.7}$$

The system (6.3.6)-(6.3.7) can be integrated numerically by using of the Runge-Kutta method, obtaining the shape of the field $n(y)$.

In the following we discuss a few numerical results. In figure 6.1 are plotted the normalized shapes of a 1-vortex ($\nu = 1$) for the three cases $\kappa = 0, \pm 1$. Because of the proportionality between ρ and the magnetic field B, figure 6.1 reproduces also the behavior of the field B having the same polarity in each point of the space and a toroidal configuration around the core of the vortex. This is strictly true when the CS coupling constant g is negative. For positive values of g the behavior of the magnetic field B have opposed sign to the field ρ and therefore the magnetic flux attached to each particle is opposed to the case $g < 0$. This is in agreement with the following symmetry that hold in presence of the CS term: $g \to -g$, $x^1 \leftrightarrow x^2$, $A^1 \leftrightarrow A^2$.
In figure 6.2, the electric field E_r for the same 1-vortex of figure 6.1, is plotted as a function of r. The polarity of E_r for the nontopological 1-vortex results to be always positive and directed radially.
In figure 6.3 and 6.4 the shape of the fields ρ and E_r, in the case of a 2-vortex ($\nu = 2$) are plotted. Now the field B has an annular shape around the core of the vortex and vanishes at the origin and the infinity. As a consequence, the electric field E_r takes a polarity inversion corresponding to the points of maximum of the field B and it is distributed in two annular concentric regions of opposite polarity placed around the core of the vortex. All the curves show the effect of the EIP consisting in a main localization ($\kappa > 0$) or delocalization ($\kappa < 0$) of the vortex. The same effect can be observed also in the shape of the electric field. Moreover, the intensity of the maximum of the field E_r is emphasized in the attractive system respect to the repulsive one, in agreement to the inclusion or exclusion effect produced by the presence of EIP. Finally, in figure 6.5 it is shown the behavior of the function $\xi_{\nu,\kappa}$ for an 1-vortex in the three cases $\kappa = 0, \pm 1$ as a function of the value of the field $\rho(r)$ at the origin, here indicated as $\rho_{\rm max}$, which we assume as an initial condition to integrate Eq. (6.3.3) (the other condition is given by $d\,\rho/d\,r = 0$ for $r = 0$). We note that in the case $\kappa = 0$ the function $\xi_{1,0}$ does not depend on $\rho_{\rm max}$ and its value is 1. This result, obtained numerically, is equal to the one derived analytically within the model of Jackiw and Pi, and implies the discretization of the electric charge. On the contrary, in the case $\kappa = \pm 1$ the quantity $\xi_{1,0}$ is a continuous increasing ($\nu = -1$) or decreasing ($\nu = 1$) monotonic function of $\rho_{\rm max}$.
This dependence of $\xi_{1,\pm 1}$ on the initial condition implies that now Q and $< M >$ lose their discretization and become continuous quantities. In fact, the value of the charge and of the other quantities depend from the asymptotic behavior of the fields around the zeros of ρ which turn out to be function of the integration constant. Now, the requirement that the field ψ should be a single value forces the constant C_0 to be an integer, but no condition are met for C_∞.
In presence of the EIP we can observe a new fact. From figure 6.5 we note that when

$\kappa = -1$ the parameter $\xi_{_{1,-1}}$ and than also Q and $< M >$ have an upper bound. So, as for other nonanalitically integrable models we have obtained continuous quantities but for a repulsive system these quantities can run in a limited range. Numerically we obtain $\xi_{_{1,-1}}^{\max} \simeq 1.156$.

In conclusion, we have studied the static solutions of a model describing a many body system in the mean field approximation, obeying to a generalized exclusion-inclusion principle and in the presence of the Chern-Simons interaction. By selecting the nonlinear potential $\widetilde{U}(\rho)$ in the form (6.1.20) we have shown the existence of self-dual static solutions, satisfying a nonlinear first order differential equation *á la Bogomol'nyi*. The solutions are states with zero energy and linear momentum. Subsequently, we have considered the subset of the vortex-like solutions, obtaining the expression of the electric charge and the angular momentum, while their shape has been determinate by numerical integration of Eq. (6.3.3). The model here studied can be considered as a continuum deformation of the Jackiw and Pi one [51, 115] performed by the parameter κ introduced by the exclusion-inclusion principle. In the model of Jackiw and Pi the vortex solution takes discrete values for the charge, magnetic flux and angular momentum which are proportional to the vorticity number. In the present model these quantities take values in the continuum and when the system obeys to an exclusion principle ($\kappa < 0$) it has an upper bound.
Finally, we remark that in the model of Jackiw and Pi, dynamical solutions can be obtained from the static ones by a boosting, as a consequence of its invariance respect to the Galilei symmetry. On the other hand, in the frame of our model, as we have shown in chapter III, the EIP potential breaks this symmetry, so that we must specifically study nonstatic solutions.

Chapter VII

Conclusions

In this work a nonlinear canonical theory describing systems of identical particles obeying to a generalized exclusion-inclusion principle (EIP) has been developed. In the mean field approximation EIP takes into account collective effects of repulsive (exclusion) or attractive (inclusion) character.
The system here considered is described by a nonlinear Schrödinger equation obtained, in the picture of the canonical quantization, from a classical model obeying to EIP, requiring that the quantum current density in the continuity equation takes the expression $\boldsymbol{j}_\kappa = \boldsymbol{j}_0\,(1+\kappa\,\rho)$ where $\boldsymbol{j}_0$ is the standard quantum current density of the linear theory.
To this purpose we have introduced in the evolution equation a complex nonlinear term $\Lambda(\rho,\,\boldsymbol{j})$ deriving from a nonlinear potential $U(\psi,\,\psi^*)$ which simulates a collective effect among particles. The expression of this nonlinearity is strongly affected by the quantization method adopted. In fact, the kinetic approach here used fixes only the imaginary part of $\Lambda(\rho,\,\mathbf{j})$. The real part is determined by the requirement that the system be canonical, which leads to an expression for $\mathrm{Re}\Lambda(\rho,\,\boldsymbol{j})$ defined modulo of a quantity obtainable from an arbitrary real potential $U(\rho)$. This arbitrarily allows to consider within EIP other interactions acting among the particles.
After the introduction of the Lagrangian and Hamiltonian functions, we have taken into account the Ehrenfest relations. From their analysis we obtain that the canonical systems obeying EIP, also in the presence of an arbitrary potential $U(\rho)$ and in the absence of external forces, are non dissipative processes (E, $<P>$ and $<M>$ are conserved) and obey to a nonlinear kinetic ($d<\boldsymbol{x}_c>/dt = \int \boldsymbol{j}_0\,(1+\kappa\,\rho)\,dx$ where $<\boldsymbol{x}_c>$ is the mean value of the central mass of the system). Thus, EIP potential $U_{\rm EIP}$ does not modify the dynamic behavior which can be affected only by the presence of an external potential V. Conversely, $U_{\rm EIP}$ is responsible for the formation of localized stationary states (solitons).

These properties of the nonlinear EIP potential are obtained rigorously studying the symmetries of Schrödinger equation. By studying the space-time translations and using the Nöther theorem, the expression of the energy-momentum tensor and the related conserved quantities have been obtained. The main results are:

1. EIP changes the expression of the energy density T^{00} and of the fluxes T^{i0} and T^{ij}. The energy density is a semidefinite positive quantity both for positive and negative value of κ, consistently with a nonrelativistic theory.

2. The expression of the momentum density is different from that of the current ($\boldsymbol{P} \neq m\,\boldsymbol{j}$) and, as a consequence, the Galilei invariance is lost.

3. The presence of $U_{\rm EIP}$ introduces in the system a dimensional coupling constant ($[\kappa] = L^{-D}$) and, as a consequence, the conformal symmetry of the system is lost.

4. We have found a scaling transformation which permits us to reduce the coupling constant κ to 1 when the inclusion principle holds, or to -1 when the exclusion principle holds.

5. The discretized symmetries P and T are not lost in the presence of $U_{\rm EIP}$.

A powerful issue, for its generality, is obtained introducing a class of nonlinear unitary transformations. In the Schrödinger equation the nonlinear term introduced by EIP is a complex one. By means of an appropriate transformation, it is possible to make real this quantity. As a consequence, the transformed system, which in general is described again by a nonlinear Schrödinger equation, obeys to a linear continuity equation where the quantum current is the same as in the linear theory. This method, introduced by us for EIP systems, can be applied to a large class of nonlinear Schrödinger equations obtained from a variational principle.
As an application of the model describing the dynamics of a collective neutral quantum particles obeying to EIP we have studied the property of solitary wave solutions. These results can be used to study Bose-Einstein condensates recently obtained at low temperature in rarefied vapor of metals like ^{7}Li, ^{23}Na and ^{87}Rb. Typically these structures are studied by means of the cubic nonlinear Schrödinger equation. In place of this nonlinearity, other nonlinear terms can be considered like, for instance, those introduced by EIP with positive value of the κ parameter to take into account the attractive effects due to the statistical interaction.
We have studied systems subjected to EIP where the matter field is coupled to a gauge field of the abelian group $U(1)$.
We have also studied a theoretical model in the presence of gauge fields with a

dynamics described by a Maxwell-Chern-Simons Lagrangian. The most important properties of the system and its symmetries have been studied. We have shown that the anyonic behavior due to the Chern-Simons term is not disturbed by the presence of EIP. Thus the system describes anyons obeying to EIP in the configuration space.

Applications of this model can be found in the physics of condensed matter and in the high T_c superconductors models. We have investigated stationary solutions when the dynamics of the gauge fields is described by the Chern-Simons term alone. In presence of the potential U_{EIP} and of a nonlinear potential, real function of the field ρ, we have found self-dual solutions describing N-vortex systems.

The principal properties of these excited states are analytically studied. After a numerical analysis, we can draw the shape of the vortices and compare them to the ones already known in literature and deduced without considering EIP. The conclusive results are:

1. The values of the charge and angular momentum of the N-vortices are continuous, while in absence of EIP are discrete.

2. In case of systems obeying to an exclusion principle the value of these observables hase an upper bound and the limits can be calculated numerically.

The numerical analysis permits us to relate the vortex profiles of systems with both the exclusive and inclusive effects of different weight showing how the condensate amplitude changes when the intensity of EIP changes.

Bibliography

[1] H. S. Green, Phys. Rev. **90**, *270 (1953).*

[2] Y. Chnuki, and S. Kamefuchi, *Quantum Field Theory and Parastatistics,* (Springer-Verlag, New York, 1982).

[3] G. F. Dell'Antonio, O. W. Greenberg, and E. C. G. Sudarshan, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gürsey (Gordon and Breach, New York 1964), p. 403.

[4] A. B. Govorkov, Physica A **203**, *655 (1993).*

[5] O. W. Greenberg, Phys. Rev. Lett. **13**, *698 (1964).*

[6] A. Y. Ignatev, and V. A. Kuzmin, Sov. J. Nucl. Phys. **46**, *444 (1987).*

[7] L. C. Biedenharn, P. Truini, and H. van Dam, J. Phys. A **22** *l67 (1989).*

[8] O. W. Greenberg, and R. N. Mohapatra, Phys. Rev. D **39**, *2032 (1989).*

[9] S. Doplicher, R. Haag, and J. Roberts, Commun. Math. Phys. **35**, *49 (1974).*

[10] K. Fredenhagen, Commun. Math. Phys. **79**, *141 (1981).*

[11] O. W. Greenberg, Phys. Rev. Lett. **64**, *705 (1990);* Phys. Rev. D **43**, *4111 (1991).*

[12] R. N. Mohapatra, Phys. Lett. **242B**, *407 (1990).*

[13] O. W. Greenberg, Physica A **180**, *419 (1992).*

[14] O. W. Greenberg, and R. C. Hilborn, Phys. Rev. Lett. **83**, *4460 (1999).*

[15] M. Chaichian, R. Gonzalez Felipe, and C. Montonen, J. Phys. A: Math. Gen. **26**, *4017 (1993).*

[16] H. S. Song, S. X Ding, and I. An, J. Phys. A: Math. Gen. **26**, *5197 (1993).*

[17] M. R. Ubriaco, Phy. Rev. E **55**, *291 (1997).*

[18] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtajan, Algebra i Analys **1**, *178 (1989).*

[19] F. Wilczek, Phys. Rev. Lett. **48**, *114 (1982);* Phys. Rev. Lett. **49**, *957 (1982).*

[20] F. Wilczek, *Fractional statistics and Anyon Superconductivity,* (World Scientific, Singapore 1990).

[21] E. Artin, Abh. Math. Sem. Hamburg **4**, *47 (1926);* Annals of Math. **48**, *101 (1947).*

[22] G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Math. Phys. **21**, *650 (1980).*

[23] R. B. Langhulin, Phys. Rev. **B23**, *3383 (1983).*

[24] D. P. Arovas, J. R. Schrieffer, and F. Wilzek, Phys. Rev. Lett. **53**, *772 (1984).*

[25] B. I. Halperin, Phys. Rev. Lett. **52**, *1583 (1984).*

[26] Y. H. Chen, F. Wilczek, E. Witten, and B. I. Halperin, Int. Jour. Mod. Phys. **B3**, *1001 (1989).*

[27] Y. S. Wu, Phys. Rev. Lett. **52**, *2103 (1984).*

[28] A. Lerda, *Anyons: Quantum Mechanics of Particles with Fractional Statistic,* Lecture note in Physics Vol. 14, (Berlin, Springer 1992).

[29] D. P. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. **B251**, *117 (1985).*

[30] T. H. Hansson, M. Roček, I. Zahed, and S. C. Zhang, Phys. Lett. **214B**, *475 (1988).*

[31] F. D. M. Haldane, Phys. Rev. Lett. **67**, *937 (1991).*

[32] Y. S. Wu, Phys. Rev. Lett. **73**, *922 (1994).*

[33] F. Calogero, J. Math. Phys. **10**, *2191 (1969);* **10**, *2197 (1969).*

[34] B. Sutherland, J. Math. Phys. **12**, *246 (1971);* **12**, *251 (1971);* Phys. Rev. A **4**, *2019 (1971);* **5**, *1372 (1972).*

[35] D. Sen, and R. K. Bhaduri, Phys. Rev. Lett. **74**, *3912 (1995).*

[36] M. V. N. Murthy, and R. Shankar, Phys. Rev. Lett. **72**, *3912 (1995).*

[37] A. D. de Veigy, and S. Ouvry, Phys. Rev. Lett. **72**, *600 (1994);* Mod. Phys. Lett. **B9**, *271 (1995).*

[38] G. E. Uhlenbeck, and L. Gropper, Phys. Rev. **41**, *79 (1932).*

[39] E. P. Gross, Nuovo Cim. **20**, *454 (1961);* J. Math. Phys. **4**, *195 (1963).*

[40] I. Barashenkov, and A. Harin, Phys. Rev. Lett. **72**, *1575 (1994).*

[41] S. Stringari, Phys. Rev. Lett. **77**, *2360 (1996).*

[42] M. Holland, and J. Cooper, Phys. Rev. A **53**, *R1954 (1996).*

[43] M. Edwards, R. J. Dodd, C. W. Clark, P. A. Ruprecht, and K. Burnett, Phys. Rev. A **53**, *R1950 (1996).*

[44] P. A. Ruprecht, M. J. Holland, and K. Burnett, Phys. Rev. A **51**, *4704 (1995).*

[45] A. Smerzi, and S. Fantoni, Phys. Rev. Lett. **78**, *3589 (1997).*

[46] I. A. Ivonin, Zh. Éksp. Teor. Fiz. **40**, *1233 (1997)* [Sov. Phys. JETP **85**, *6 (1997)*].

[47] N. Papanicolaou, and T. N. Tomaras, Phys. Lett. **179**, *33 (1993).*

[48] V. Ginsburg, and L. Pitayevski, Zh. Éksp. Teor. Fiz. **34**, *1240 (1958)* [Sov. Phys. JETP **7**, *858 (1958)*].

[49] L. Pitayevski, Zh. Éksp. Teor. Fiz. **34**, *646 (1961)* [Sov. Phys. JETP **13**, *451 (1961)*].

[50] A. Kovalev, and A. Kosevich, Fiz. Nizk. Temp. **2**, *913 (1976).*

[51] R. Jackiw, and S. Y. Pi, Phys. Rev. D **42**, *3500 (1990).*

[52] M. Ablowitz, and S. De Lillo, Nonlinearity **7**, *1143 (1994).*

[53] H. D. Doebner, and G. A. Goldin, Phys. Lett. A **162**, *397 (1992).*

[54] H. D. Doebner, and G. A. Goldin, Phys. Rev. A **54**, *3764 (1996).*

[55] U. Aglietti, L. Griguolo, R. Jackiw, S. Y. Pi, and D. Seminara, Phys. Rev. Lett. **77**, *4406 (1996).*

[56] R. Y. Chiao, E. Garmire, and C. H. Townes, Phys. Rev. Lett. **13**, *479 (1964).*

[57] V. I. Talanov, Pisma v. Zh. Éksp. Teor. Fiz. **2**, *222 (1965)* [JETP Lett. **2**, *138 (1965)*].

[58] P. L. Kelley, Phys. Rev. Lett. **15**, *1005 (1965).*

[59] V. E. Zakharov, and Shabat, Zh. Éksp. Teor. Fiz. **61**, *118 (1971).*

[60] K. Lonngren, and A. Scott, *Solitons in Action, (Accademic Press, New York 1981).*

[61] R. Rajaraman, *Solitons and Instantons,* (Elsevier, North Holland 1982).

[62] G. 'tHooft, Nucl. Phys. **B79**, *276 (1974).*

[63] A. M. Polyakov, Pisma v. Zh. Éksp. Teor. Fiz. **20**, *430 (1974)* [JETP Lett. **20**, *194 (1974)*].

[64] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura. Phys. Rev. Lett. **19**, *1095 (1967).*

[65] P. J. Olver, *Applications of Lie Groups to Differential Equations,* (Springer, New York, 1986).

[66] F. Calogero, and S. De Lillo, Inverse Problem **3**, *633 (1987).*

[67] G. Kaniadakis, and P. Quarati, Phys. Rev. E **48**, *4263 (1993).*

[68] G. Kaniadakis, and P. Quarati, Phys. Rew. E **49**, *5103 (1994).*

[69] G. Kaniadakis, Phys. Rew. E **49**, *5111 (1994).*

[70] G. Kaniadakis, Phys. Lett. A **203**, *229 (1995).*

[71] G. Kaniadakis, A. Lavagno, and P. Quarati, Nucl. Phys. B **466**, *527 (1996).*

[72] G. Kaniadakis, A. Lavagno, and P. Quarati, Mod. Phys. Lett. B **10**, *497 (1996).*

[73] E. Nelson, Phys. Rev. **150**, *107 (1966).*

[74] I. Fenyes, Z. Phys. **132**, *82 (1952).*

[75] W. Weizel, Z. Phys. **134**, *264 (1953);* **136**, *582 (1954).*

[76] G. Kaniadakis, Phys. Lett. A **206**, *229 (1994).*

[77] G. Kaniadakis, Phys. Rev. A **55**, *941 (1996).*

[78] A. Lavagno, G. Kaniadakis, and P. Quarati, Z. Phys. A **355**, *185 (1996).*

[79] I. M. Gel'fand, and S. V. Fomin, *Calculus of Variations,* (Prentice-Hall, Englewood Cliffs, New York 1963).

[80] H. Goldstein, *Classical Mechanics,* (Addison-Wesley, Reading, Mass. 1950).

[81] P. A. M. Dirac, *Lectures on Quantum Mechanics,* No. 2 of Belfer Graduate School of Science Monography Series (Yeshiva University, New York, 1964).

[82] E. Nöther, *Invariante Variationsproblem,* Nachr. d. König. Gesellsh. d. Wiss. zu Göttingen, Math-phys. Klasse, *235 (1918);* English translation M. A. Travel, Transport Theory and Statistical Physics *1(3), 183 (1971).*

[83] D. Bohm, Phys. Rev. **85**, *166 (1951).*

[84] E. Madelung, Z. Phys. **40**, *332 (1926).*

[85] R. W. Hasse, Phys. Rev. A **25**, *583 (1982).*

[86] M. A. de Moura, Phys. Rev. A **37**, *4998 (1988).*

[87] M. D. Kostin, J. Chem. Phys. **57**, *3589 (1972).*

[88] D. Schuch, Phys. Rev. A **55**, *935 (1997).*

[89] R. M. Miura, C. S. Garden, and M. D. Kruskal, Math. Phys. **9**, *1204 (1968).*

[90] V. E. Zakharov, and A. B. Shabat, Sov. Phys. JETP **34**, *62 (1972).*

[91] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. **53**, *249 (1974).*

[92] D. J. Kaup, and A. C. Newell, J. Math. Phys. **19**, *798 (1978).*

[93] F. Calogero, A. Degasperis, and S. De Lillo, J. Phys. A: Math. Gen. **30**, *5805 (1997).*

[94] F. Calogero, J. Phys. A **34**, *3197 (1993).*

[95] H. D. Doebner, and G. A. Goldin, J. Phys. A **27**, *1771 (1994).*

[96] C. R. Hagen, Phys. Rev. D **5**, *377 (1972).*

[97] R. P. Feynman, and A. R. Hibbs, *Quantum Mechanics and Path Integrals,* McGraw-Hill, New York, 1965.

[98] F. Guerra, and M. Pusterla, Lett. Nuovo Cimento **34**, *351 (1982).*

[99] V. L. Ginzburg, and L. D. Landau, Zh. Éksp. Teor. Fiz. **20**, *1064 (1950).*

[100] L. P. Pitayevskii, Zh. Éksp. Teor. Fiz. **40**, *646 (1961)* [Sov. Phys. JETP **13**, *451 (1961)*].

[101] G. N. Stratopoulos, and T. N. Tomaras, Phys. Rev. B **54**, *493 (1996).*

[102] H. G. Kaper, and P. Takáč, Nonlinearity **11**, *291 (1998).*

[103] R. B. Laughlin, Phys. Rev. Lett. **50**, *1395 (1983).*

[104] A. Khurana, Phys. Today **43** *(***1***), 19 (1990).*

[105] S. Weinberg, *Understanding the Fundamental Constitutents of Matter* (A. Zichichi editor, Plenum Press, New York 1978).

[106] J. Fröhlich and A. Zee, Nucl. Phys. B **364**, *517 (1991).*

[107] K. B. Lyons, J. Kwo, J. F. Dillion, G. P. Espinosa, M. McGlashan-Powell, A. Ramirez, and L. Schneemeyer, Phys. Rev. Lett. **64**, *2949 (1990).*

[108] H. J. Weber, D. Weitbrecht, D. Brach, A. L. Shelankov, H. Keiter, W. Weber, T. Wolf, J. Geerk, G. Linker, G. Roth, P. C. Splittgerber-Hünnekes, and G. Güntherodt, Solid State Commun. **76**, *511 (1990).*

[109] W. Siegel, Nucl. Phys. **B156**, *135 (1979).*

[110] J. Schonfeld, Nucl. Phys. **B185**, *157 (1981).*

[111] R. Jackiw, and S. Templeton, Phys. Rev. D **23**, *2291 (1981).*

[112] C. R. Hagen, Ann. Phys. (N.Y.) **157**, *342 (1984).*

[113] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, *975 (1982);* Ann. Phys. (N.Y.) **140**, *372 (1982).*

[114] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, Phys. Rep. **209**, *199 (1991).*

[115] R. Jackiw, and S. Y. Pi, Phys. Rev. Lett. **64**, *2969 (1990);* Phys. Rev. Lett. **66**, *2682 (1991);* Prog. Theor. Phys. Suppl. **107**,*1 (1992).*

[116] Y. Aharonov, and D. Bohm, Phys. Rev. **115**, *485 (1959).*

[117] S. Sinha, Phys. Rev. A **55**, *4325 (1997).*

[118] W. P. Reinhardt, and C. W. Clark, J. Phys. B **30**, *L785 (1997).*

[119] V. I. Yukalov, E. P. Yukalova, and V. S. Bagnato, Phys. Rev. A **56**, *4845 (1997).*

[120] S. A. Morgan, R. J. Ballagh, and K. Burnett, Phys. Rev. A **55**, *4338 (1997).*

[121] R. W. Hasse, Z. Phys. B **37**, *83 (1980).*

[122] A.P. Polychronakos, Phys. Rev. Lett. **74**, *5153 (1995).*

[123] V. L. Ginzburg, and L. P.Pitayevski, Zh. Éksp. Teor. Fiz. **34**, *1240 (1958)* [Sov. Phys. JETP **7**, *858 (1958)*].

[124] J. R. Gardner, R. A. Cline, J. D. Miller, D. J. Heinzen, H. M. J. M. Boesten, and B. J. Verhaar, Phys. Rev. Lett. **74**, *3764 (1995).*

[125] A. A. Abrikosov, Zh. Éksp. Teor. Fiz. **32**, *956 (1957)* [Sov. Phys. JETP **5**, *1174 (1957)*].

[126] E. B. Bogomol'nyi, Yad. Fiz. **24**, *861 (1976)* [Sov. J. Nucl. Phys. **24**, *449 (1976)*].

[127] H. B. Nielsen, and P. Olesen, Nucl. Phys. **B61**, *45 (1973).*

[128] J. Hong; Y. Kim and P. Y. Pac, Phys. Rev. Lett. **64**, *2230 (1990).*

[129] H. J. de Vega and F. A. Schaposnik, Phys. Rev. D **14**, *1100 (1976),* Phys. Rev. Lett. **56**, *2564 (1986).*

[130] E. J. Weinberg, Phys. Rev. D **19**, *3008 (1979).*

[131] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and other Topological Defects,* (Cambridge Univ. Press, Cambridge 1994).

[132] B. Julia, and A. Zee, Phys. Rev. **D11**, *2227 (1975).*

[133] S. K. Paul, and A. Khare, Phys. Lett. **B174**, *420 (1986);* **B177**, *E453 (1986).*

[134] J. Frölich, and P. A. Marchetti, Comm. Math. Phys. **121**, *177 (1989).*

[135] Z. F. Ezawa, M. Hotta, and A. Iwazaki. Phys. Rev. D **44**, *452 (1991).*

[136] R. Jackiw, K. Lee, and E. J. Weinberg, Phys. Rev. D **42**, *3488 (1990).*

[137] S. Li, and R. K. Bhaduri, Phys. Rev. D **43**, *3573 (1991).*

[138] S. Hyun, J. Shin, J. H. Yee, and H. Lee, Phys. Rev. D **55**, *3900 (1997).*

[139] C. Duval, P. A. Horváthy, and L. Palla, Phys. Rev. D **52**, *4700 (1995).*

[140] M. Torres, Phys. Rev. D **46**, *R2295 (1992).*

[141] T. Lee, and H. Min, Phys. Rev. D **50**, *7738 (1994).*

[142] P. K. Ghosh, Phys. Rev. D **49**, *5458 (1994).*

[143] A. Antillón, J. Escalona, and M. Torres, Phys. Rev. D **55**, *6327 (1997).*

[144] G. V. Dunne, *Aspects of Chern-Simons Theory, Presented on Summer School on Topological Aspects of Low-dimensional Systems Les Houches, France (1998); preprint het-ph/9902115.*

[145] J. Liouville, J. Math. Pures Appl. **18**, *71 (1853).*

Author's Publications

1. G. Kaniadakis, P. Quarati and A. M. Scarfone, Phys. Rev. E **58**, 5574 (1998).
2. G. Kaniadakis, P. Quarati and A. M. Scarfone, Physica A **255**, 474 (1998).
3. G. Kaniadakis, P. Quarati and A. M. Scarfone, Rep. Math. Phys. **44**, 121 (1999).
4. G. Kaniadakis, P. Quarati and A. M. Scarfone, Rep. Math. Phys. **44**, 127 (1999).
5. G. Kaniadakis, and A. M. Scarfone, *Nonlinear gauge transformation for a class of Schrödinger equations containing complex nonlinearities*, Rep. Math. Phys. in press.
6. G. Kaniadakis, A. Lavagno, P. Quarati, and A. M. Scarfone, *Nonlinear gauge transformation of a quantum system obeying an exclusion-inclusion principle*, J. Nonlin. Math. Phys. in press.
7. G. Kaniadakis, and A. M. Scarfone, *A Maxwell-Chern-Simons model for a quantum system obeying an exclusion-inclusion principle*, submitted
8. G. Kaniadakis, and A. M. Scarfone, *Chern-Simons vortices in particle systems obeying an exclusion-inclusion principle*, submitted
9. G. Kaniadakis, P. Quarati, A. M. Scarfone, *Canonical quantum systems obeying an exclusion-inclusion principle,* Electronic Proc. of 11th International Workshop on NEEDS (Nonlinear Evolution Equations and Dynamical Systems), June 1997, Kolymbari-Chania, Crete, Greece.

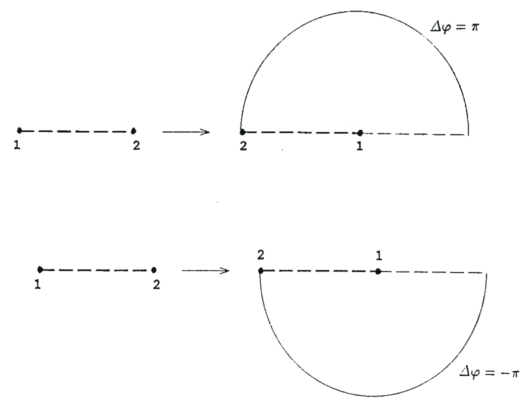

Figure 1.1: The exchange of two particle is realized by means of rotation of one around the other by an angle of $\Delta\varphi = \pm\,\pi$

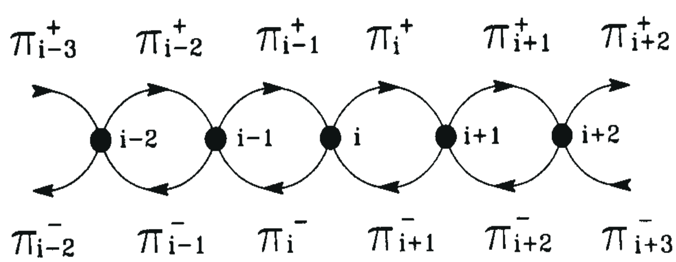

Figure 1.2: One-dimensional discrete Markoffian chain where the transition probabilities are indicated.

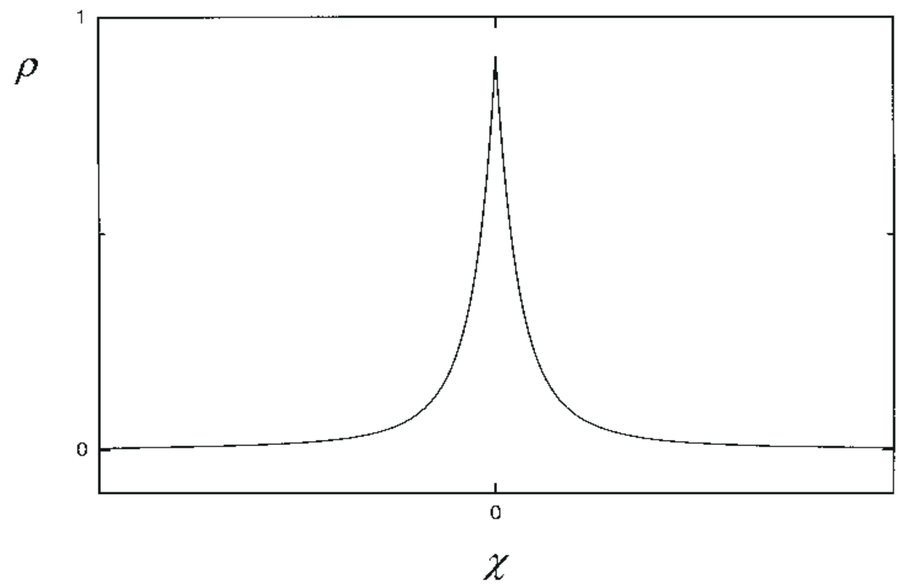

Figure 5.1: Shape of soliton (in arbitrary units) with $\alpha = 0$, $\beta = -1$ and $\sigma = 1$.

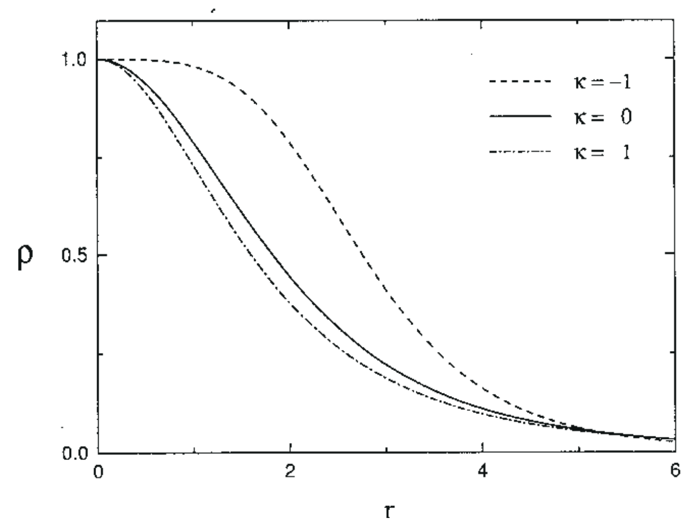

Figure 6.1: Plot of ρ for 1-vortex with $\kappa = 0, \pm 1$ versus r [in $r_0 = (\sqrt{2\,\alpha})^{-1}$ units].

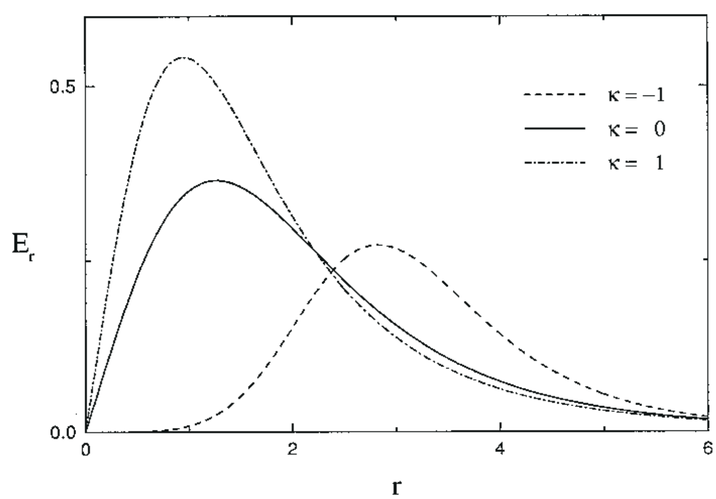

Figure 6.2: Plot of the electric field (in $E_{0_r} = \hbar^2\,\alpha^{3/2}/e\,m\,\sqrt{2}$ units) for 1-vortex with $\kappa = 0, \pm 1$ versus r [in $r_0 = (\sqrt{2\,\alpha})^{-1}$ units].

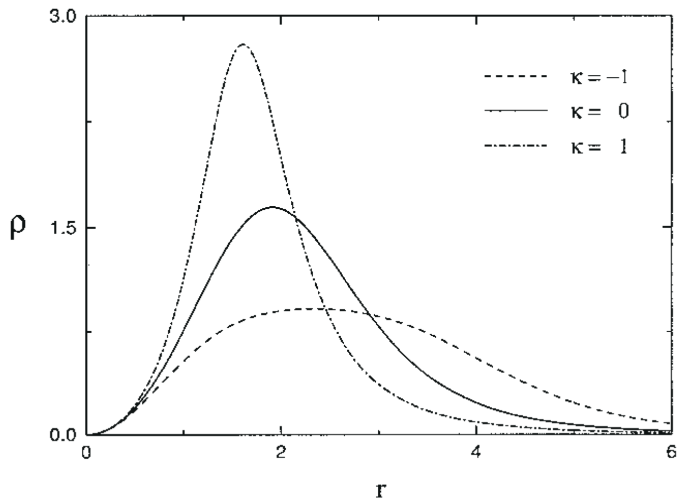

Figure 6.3: Plot of ρ for 2-vortex with $\kappa = 0, \pm 1$ versus r [in $r_0 = (\sqrt{2\,\alpha})^{-1}$ units].

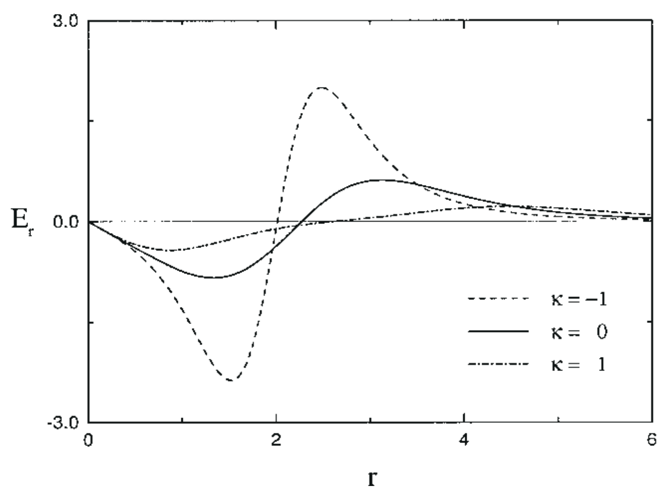

Figure 6.4: Plot of the electric field (in $E_{0_r} = \hbar^2\,\alpha^{3/2}/e\,m\,\sqrt{2}$ units) for 2-vortex with $\kappa = 0, \pm 1$ versus r [in $r_0 = (\sqrt{2\,\alpha})^{-1}$ units].

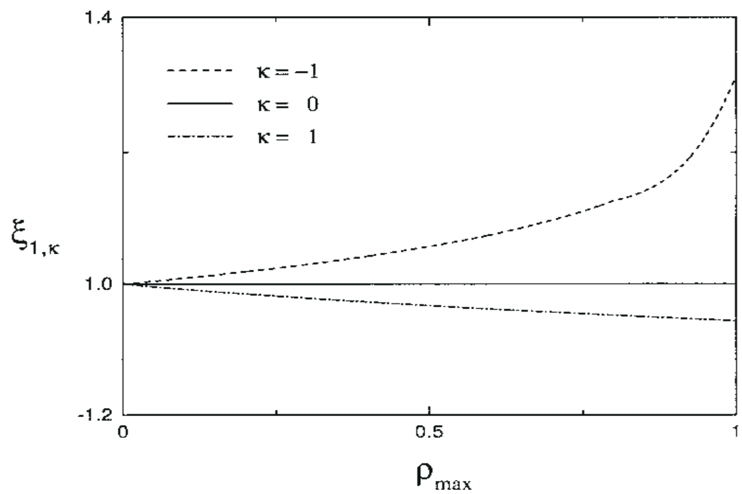

Figure 6.5: Plot of the parameter $\xi_{1,\kappa}$ versus $\rho_{\max}$ with $\kappa = 0, \pm1$.